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The Instability of the Thin Vortex Ring of Constant Vorticity

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THE INSTABILITY OF THE THIN VORTEX RING OF CONSTANT VORTICITY

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[Plate 1]

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A theoretical investigation of the instability of a vortex ring to short azimuthal bending waves is presented. The theory considers only the stability of a thin vortex ring with a core of constant vorticity (constant ζ/r) in an ideal fluid. Both the mean flow and the disturbance flow are found as an asymptotic solution in $\epsilon = a/R$, the ratio of core radius to ring radius. Only terms linear in wave amplitude are retained in the stability analysis. The solution to $O(\epsilon^2)$ is presented, although the details of the stability analysis are carried through completely only for a special class of bending waves that are known to be unstable on a line filament in the presence of strain (Tsai & Widnall 1976) and have been identified in the simple model of Widnall, Bliss & Tsai (1974) as a likely mode of instability for the vortex ring: these occur at certain critical wavenumbers for which waves on a line filament of the same vorticity distribution would not rotate ($\omega_0 = 0$). The ring is found to be always unstable for at least the lowest two critical wavenumbers ($ka = 2.5$ and 4.35). The amplification rate and wavenumber predicted by the theory are found to be in good agreement with available experimental results.

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1. Introduction

There now exists a considerable body of experimental data that indicates that laminar vortex rings at moderate Reynolds numbers are unstable to azimuthal bending waves around the perimeter. Figure 1, plate 1, taken from Widnall & Sullivan (1973), shows the general features of this instability. To the casual observer of vortex rings, this may come as a surprise, since vortex rings are generally considered to be one of the most stable and persistent flows in our common experience. In actual fact, vortex rings are stable only at low Reynolds numbers and apparently at high Reynolds numbers when the vortex cores are fully turbulent.

Because of the work, both theoretical and experimental, done over the past few years, our understanding of these flows goes well beyond the early statements of Kelvin, 'the known phenomena of ... smoke rings ... convinces ... us ... that the steady configuration ... is stable' and 'the vortex ring in an ideal fluid is indestructible' (Thomson 1867), but falls far short of a complete understanding of the behaviour of a vortex ring in a real fluid.

In our discussion of the stability of the vortex ring we will only mention but not attempt to deal with the fact that so little is known about the process of generation and roll-up. At present the vorticity distribution resulting from the formation processes cannot be predicted; this distribution is, of course, required for a complete analysis of the stability of a vortex ring. Whether all of the experimental investigators who are making vortex rings by various processes are studying the same flow is surely in doubt. Widnall & Sullivan (1973) presented the first measurements of the vorticity distribution, within a particular ring, using an l.d.v.; Maxworthy (1976) has recently reported similar measurements, but no theory exists to predict the vorticity distribution resulting from a given process. Both of these experiments showed that vorticity is neither constant within the ring nor confined within a core of small diameter, but it may be sufficiently concentrated so that a small core of constant vorticity may be used as a model for the flow.

At low Reynolds numbers (say, below $Re \simeq 600$ based on propagation velocity and ring radius), stable vortex rings are formed (Maxworthy 1972). For higher Reynolds numbers, if the ring is laminar after the process of generation and roll-up, unstable azimuthal waves develop and grow until finite-amplitude breaking occurs. Out of this process, a turbulent vortex ring is formed whose properties are not well understood. (See, however, Maxworthy (1976) for a recent experimental study of the properties of this flow.) This turbulent vortex ring is apparently stable, whereas a laminar ring of the same mean vorticity distribution would probably be unstable. Maxworthy's experiments also indicate that the turbulent vortex ring, under some conditions, has axial velocities within the core, a feature that is not present in its laminar counterpart. The effects of these axial flows on the stability of the vortex ring are not currently understood.

The by-now-well-established phenomenon of azimuthal wave instabilities on vortex rings was described qualitatively by Maxworthy (1972) in a paper concerned primarily with the structure of stable (low Re) rings. This phenomenon was characterized quantitatively in the experiments of Widnall & Sullivan (1973) in which measurements were made of the vorticity distribution, the self-induced velocity and circulation, and the amplification rate for several different rings. A complete set of measurements was obtained only for the fattest ring studied. Both of these works were preceded by the flow visualization studies of Krutzsch (1939) whose photographs of unstable azimuthal waviness had evidently been overlooked or unappreciated, since there was apparently no follow-up to his observations.

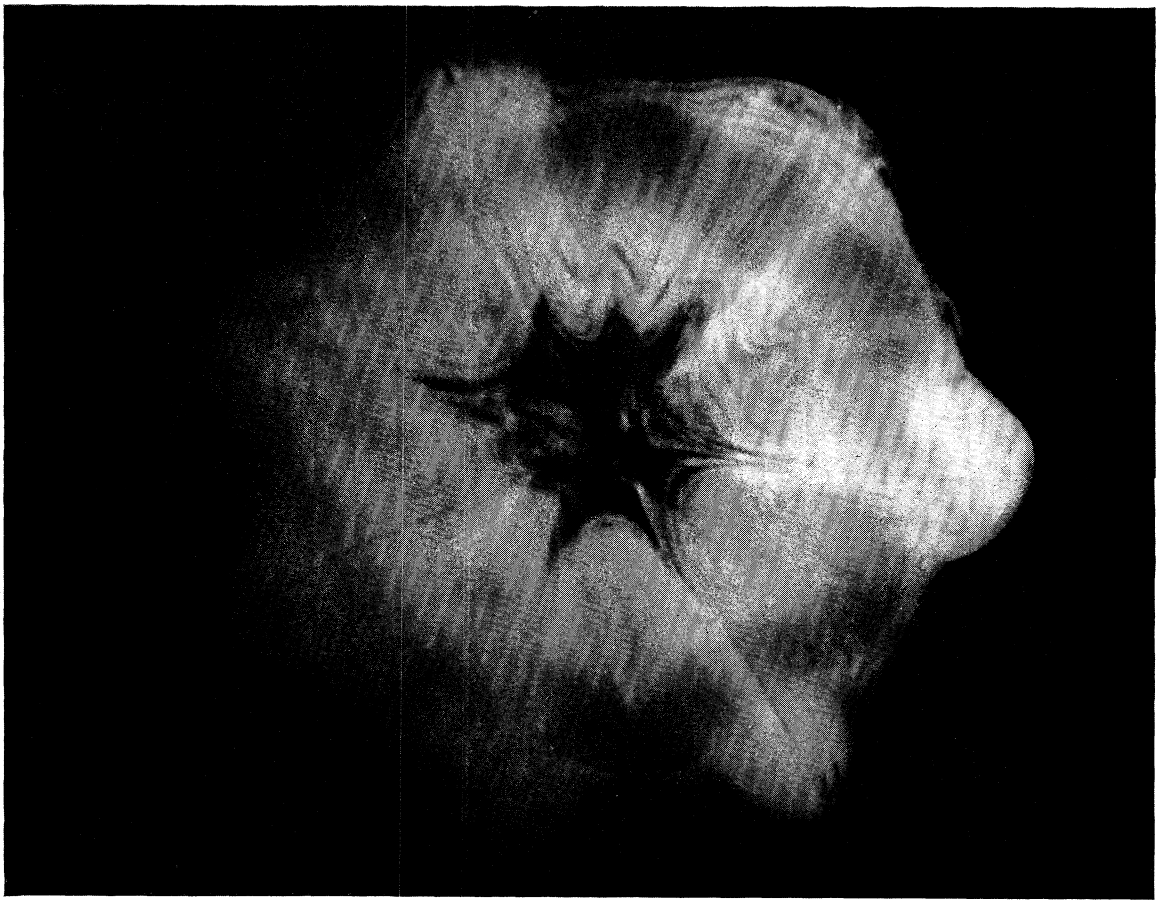

FIGURE 1. Flow visualization of the vortex ring instability; $n = 7$. Taken from Widnall & Sullivan (1973).

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(*Facing p.* 274)

Although the first theoretical attempt to explain the observed instability was that of Widnall & Sullivan (1973), the related work of Kelvin (Thomson 1867), J. J. Thomson (1883) and others was directed towards proving the *stability* of the vortex ring and calculating its frequencies of vibration in order to lay the foundation for the vortex theory of atomic structure. While we do not intend to review this early work to illuminate the differences between the analysis that led to different conclusions for what is, after all, the same problem (the vibrations of a thin vortex ring with constant vorticity), we shall show in our analysis that the proper treatment of the internal structure of the flow within the vortex core due to the bending waves is crucial to the stability analysis. Thus, neither the analysis of J. J. Thomson (1883) (whose model is not adequate to include internal structure) nor that of Pockington (1895) (who considered the vibrations of a hollow vortex ring which has no fluid in the core) predicts the observed instability.

The theoretical analysis of Widnall & Sullivan (1973) was based on the previous work of Widnall, Bliss & Zalay (1971) in which a general asymptotic analysis was presented to predict the self-induced motion of a slender vortex filament with an arbitrary distribution of vorticity and axial velocity within the core. This theory requires that changes along the filament be negligible in comparison to changes over the cross-section of the core; this is valid only if the wavelength of the perturbations is large in comparison with the radius of the vortex core. This approach is successful in predicting the long-wave instability of a vortex pair where the wavelength is about eight times the spacing between the filaments and allows the effects of non-uniform distributions of vorticity to be easily incorporated into the stability analysis (Widnall *et al.* 1971).

Widnall & Sullivan (1973) applied this analysis to study the stability of azimuthal waves on a vortex ring. Unfortunately, the instability predicted by this long-wave analysis occurs at a wavelength that is too short for the analysis to be valid. However, this theoretical work was revealing in two important respects: the prediction that the vortex ring was unstable at the wavenumber for which the asymptotic analysis spuriously predicts that a sinusoidally perturbed line filament would have zero self-induced rotation ($\omega_0 = 0$) provided a significant clue to the physical mechanism responsible for the instability; and the agreement of this theory with the general features of the instability, the amplification rate and the increase in azimuthal wavenumbers with decreasing core size indicated that the instability could likely be predicted by an inviscid analysis of the sinusoidal bending perturbations of a slender vortex ring without the need to incorporate more complex features of the flow. For example, an early explanation of the instability (Maxworthy 1972) attributed it to vorticity of the opposite sign swept into the core during the rollup process.

In 1974, Widnall *et al.* presented a physically plausible but not mathematically rigorous argument that bending waves on a vortex filament would be unstable in the presence of a straining flow (such as that of the ring mean velocity field itself) whenever these waves had no self-induced rotation ($\omega_0 = 0$). The first radial modes of bending of a filament do not, in general, have the property that $\omega_0 = 0$ for some value of wavenumber but the second radial mode, in which the centre moves in a direction opposite to the boundary of the core, does have $\omega_0 = 0$ for some critical wavenumber $\kappa_2 = ka$, as do the higher radial modes which have $\omega_0 = 0$ for yet higher values of $\kappa_n = ka$. In Widnall *et al.* (1974) these critical values of wavenumber are calculated for two different vortex cores: constant and distributed vorticity. The instability of these waves in the presence of the mean strain field of the ring is then considered. This analysis is not rigorous in that the effects of ring curvature are not considered and the displaced vortex core is assumed to move with the local free stream at each section of the wave – an assumption not valid for short waves.

Despite these shortcomings, the predictions of the model agreed very well with experiment. In principle, any distribution of vorticity within the core could be considered in this model although the results indicated only a slight sensitivity to the details of the vorticity distribution.

In the present paper we present the complete analysis for the instability of a thin vortex ring as an asymptotic analysis in ϵ, the ratio of core size to ring radius. The mathematical structure of this analysis is that of an asymptotic stability problem in which the corrections to the eigenvalues at higher order are determined by solvability conditions and the removal of secular behaviour in a manner very similar to techniques of nonlinear stability analysis. In this case our expansion parameter is not the amplitude of the disturbance, but the small parameter ϵ in which the mean flow field of the ring is obtained as an asymptotic solution. The solution technique can be formulated for a thin ring of arbitrary vorticity (containing axial flows, if desired) but due to the complexities of the analysis, we here consider only vortex cores that to lowest order in ϵ have constant vorticity.

In related work motivated by the instability of the vortex ring both Moore & Saffman (1975) and Tsai & Widnall (1976) have considered the stability of a straight vortex filament in a weak strain field by expanding the perturbation solution in a small parameter ϵ, the ratio of strain to vorticity. Both analyses derive a necessary, but not sufficient, condition for instability that (in the case of a vortex filament without axial flow) is satisfied at several wavenumber–frequency combinations including those wavenumbers for which $\omega_0 = 0$ on the line filament in an undisturbed medium. Tsai & Widnall (1976) present calculations of amplification rate for a line filament without axial flow at several of the wavenumber–frequency combinations which satisfy the necessary condition for instability; the flow is unstable at some of these points, including all those values of ka examined for which $\omega_0 = 0$. These critical waves are the higher radial modes of bending on the filament that have no self-induced rotation; the vortex core does not move uniformly in bending, but various radial stations move in opposition.

The stability problem for waves on a vortex ring is more similar to that for waves on a straight filament than might be expected. As will be seen, the waves of interest for the vortex ring instability are short waves such that ka remains constant as $\epsilon \to 0$ so that ω_0 remains zero. Therefore the wavenumber k becomes large in the limit $\epsilon \to 0$. In § 3 we show that, as a result, the far-field effects from waves around the ring are asymptotically small (as $e^{-k/\epsilon}$) as $\epsilon \to 0$. Thus, only the local effects of curvature enter into the stability problem. This also means that any locally curved filament should exhibit the same instability as the vortex ring.

Moore & Saffman (1975) also considered the effect of a small axial flow on the stability of a line vortex in weak strain, concluding that the critical conditions now occur at finite ω_0; since no results are presented for amplification rate, it is not known whether axial flow stabilizes the instability of a vortex filament in the presence of strain. Consideration of axial flow is also important for the stability of the ring since there is some suggestion (Maxworthy 1976) that the finite-amplitude wave breaking of the instability waves generate axial flows along the core of the ring (it appears that these flows have no net axial momentum). If it were shown that these axial velocities stabilize the bending-wave instabilities, we would come much closer to understanding why a vortex ring with a turbulent core is apparently stable. Unfortunately, such an analysis represents a formidable and tedious task as can be verified by a glance at the remainder of this paper, in which we consider only the simplest case, a vortex ring of constant vorticity with no axial flow.

2. Formulation

We consider the linear instability of a thin vortex ring in an inviscid imcompressible flow. The vorticity is taken to be uniform within a circular core of radius a that is small in comparison to the radius R; the small parameter ϵ is taken as the ratio a/R. This is, of course, an asymptotic description of the flow; actually, $\zeta = \zeta_0\, y$ across the core and the core does not remain circular to all orders in ϵ. An asymptotic solution in ϵ is presented both for the mean flow field of the ring and for the behaviour of azimuthal bending-wave perturbations in this mean flow.

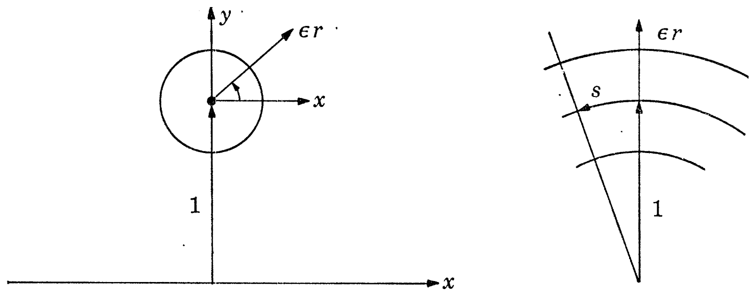

Figure 2. Local curved cylindrical coordinates: $y = 1+\epsilon r\sin\theta$; $x = \epsilon r\cos\theta$; s is the arc length around the ring at $y = 1$.

To analyse the flow in and near the vortex core we use the curved cylindrical coordinates used by Bliss (1970), in which a vortex core of circular cross-section is the thin torus $r = a$ centred on the ring radius R. The coordinates are r, θ and s, the arc length at $r = R$. When the coordinates are scaled with the ring radius R and the r coordinates then scaled with ϵ so that $r = 1$ on the core boundary, we obtain the coordinate system sketched in figure 2. The governing equations can be obtained by standard techniques (Batchelor 1967) or can be obtained by inspection by rewriting the governing equation in the x, y variables of the spherical coordinate system appropriate for the ring in a local cylindrical system centred on the vortex core so that $x = \epsilon r\cos\theta$ and $y = 1+\epsilon r\sin\theta$. (A similar coordinate system was used by Dean (1927) in his study of flow in curved pipes.) In non-dimensional form, with $\Gamma/2\pi a$ chosen as the velocity scale and $2\pi R^2/\Gamma$ chosen as the time scale, governing equations for the flow in this curvilinear coordinate system become

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial r}+\frac{V}{r}\frac{\partial U}{\partial \theta}+\frac{\epsilon W}{1+\epsilon r\sin\theta}\frac{\partial U}{\partial s}-\frac{V^2}{r}-\frac{\epsilon W^2\sin\theta}{1+\epsilon r\sin\theta} = -\frac{\partial P}{\partial r}, \tag{2.1a}$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial r}+\frac{V}{r}\frac{\partial V}{\partial \theta}+\frac{\epsilon W}{1+\epsilon r\sin\theta}\frac{\partial V}{\partial s}+\frac{UV}{r}-\frac{\epsilon W^2\cos\theta}{1+\epsilon r\sin\theta} = -\frac{1}{r}\frac{\partial P}{\partial \theta}, \tag{2.1b}$$

$$\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial r}+\frac{V}{r}\frac{\partial W}{\partial \theta}+\frac{\epsilon W}{1+\epsilon r\sin\theta}\frac{\partial W}{\partial s}+\frac{\epsilon W(U\sin\theta+V\cos\theta)}{1+\epsilon r\sin\theta} = \frac{-\epsilon}{1+\epsilon r\sin\theta}\frac{\partial p}{\partial s}, \tag{2.1c}$$

$$\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial \theta}+\frac{\epsilon}{1+\epsilon r\sin\theta}\left(U\sin\theta+V\cos\theta+\frac{\partial W}{\partial s}\right) = 0. \tag{2.1d}$$

The potential flow outside the core satisfies Laplace's equation in the form

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{\epsilon\sin\theta}{1+\epsilon r\sin\theta}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\epsilon\cos\theta}{r(1+\epsilon r\sin\theta)}\frac{\partial\Phi}{\partial\theta}+\frac{\epsilon^2}{(1+\epsilon r\sin\theta)^2}\frac{\partial^2\Phi}{\partial s^2} = 0. \tag{2.1e}$$

U, V, W, P and Φ are, respectively, the radial, swirl and axial velocity components, and the pressure and velocity potential outside the core.

We formulate the problem by expanding both the steady-flow solution for the ring field, and the unsteady wave perturbations in an asymptotic series in ϵ; only terms linear in wave amplitude will be considered.

For a vortex ring, the steady-flow asymptotic solution near the ring is of the form (Fraenkel 1972; Bliss 1973)

$$\boldsymbol{Q}(r,\theta) = \boldsymbol{Q}_0(r,\theta)+\epsilon\boldsymbol{Q}_1(r,\theta)+\epsilon^2\ln\epsilon\;\boldsymbol{Q}_{12}(r,\theta)+\epsilon^2\boldsymbol{Q}_2(r,\theta)+\dots, \tag{2.2}$$

where $\boldsymbol{Q}$ is an extended vector that includes the velocity components U, V, W, the pressure P, the velocity potential Φ (for $r>1$) and the shape of the core boundary $r=\Theta(\theta)\equiv 1+\alpha(\theta)$

$$\boldsymbol{Q}(r,\theta) = \{U(r,\theta),\,V(r,\theta),\,W(r,\theta),\,P(r,\theta),\,\Phi(r,\theta),\,\Theta(\theta)\}. \tag{2.3a}$$

$\boldsymbol{Q}_0$ is the solution for the flow field of a straight line filament ($\epsilon\equiv 0$); $\boldsymbol{Q}_1$, $\boldsymbol{Q}_{12}$ and $\boldsymbol{Q}_2$ are the higher-order terms in the mean-flow solution that represent the various effects of the curvature of the filament into a ring. For a straight line filament with constant vorticity and no axial velocity

$$\boldsymbol{Q}_0 = \{0,\,r,\,0,\,\tfrac{1}{2}r^2-1,\,\theta,\,1\}. \tag{2.3b}$$

To obtain the disturbance equations for short waves, the arc length s is also scaled by ϵ and the wave number is referred to core size a so that for short waves $k\sim O(1)$ as $\epsilon\to 0$. The disturbance flow $\tilde{\boldsymbol{q}}(r,\theta,t)$ due to waves on the ring, harmonic in time and azimuth, is then expanded in a similar asymptotic series (without loss of generality, k will be taken as positive)

$$\tilde{\boldsymbol{q}}(r,\theta,t) = \boldsymbol{q}(r,\theta)\,\mathrm{e}^{\mathrm{i}(\omega t+ks)} = \{\boldsymbol{q}_0(r,\theta)+\epsilon\boldsymbol{q}_1(r,\theta)+\epsilon^2\ln\epsilon\;\boldsymbol{q}_{12}(r,\theta)+\epsilon^2\boldsymbol{q}_2(r,\theta)\}\,\mathrm{e}^{\mathrm{i}(\omega t+ks)}, \tag{2.4}$$

where $\boldsymbol{q}$ is the disturbance vector with components velocity, pressure, potential, and displacement of the core boundary $f(\theta)$ due to the disturbance.

$$\boldsymbol{q}(r,\theta) = \{u(r,\theta),\,v(r,\theta),\,w(r,\theta),\,\pi(r,\theta),\,\phi(r,\theta),f(\theta)\}. \tag{2.5}$$

The eigenfrequency ω of the perturbations is also expanded as

$$\omega = \omega_0+\epsilon\omega_1+\epsilon^2\ln\epsilon\;\omega_{12}+\epsilon^2\omega_2+\dots. \tag{2.6}$$

In the remainder of the analysis, the $O(\epsilon^2\ln\epsilon)$ and $O(\epsilon^2)$ terms in the solution will be grouped together. The behaviour of the $\epsilon^2\ln\epsilon$ terms themselves is of some interest and is later discussed. To solve for the flow in and near the core, we will use the assumed form of solution (2.2), (2.4) and (2.6) in the governing equations (2.1) in their linearized (small-amplitude) form. This procedure will lead to the set of equations governing each term in the asymptotic solution. It will be seen that to obtain instability, the solution must be determined to $O(\epsilon^2)$.

In the region away from the core, this procedure, of course, cannot be followed since ϵr is no longer a small quantity. In this region, the flow satisfies the three-dimensional Laplace's equation in outer (unscaled by ϵ) variables. Laplace's equation does not separate in our convenient local curvilinear coordinate system, but it does separate in the related toroidal coordinate system (S, η, ψ) which can be chosen so that the core boundary is a coordinate surface $S=$ constant, as sketched in figure 3. Therefore, the solution well outside the core is found as an expansion in toroidal functions.

$$\phi = (S-\cos\eta)^{\frac{1}{2}} \sum_m \sum_n A_n^m P_{m-\frac{1}{2}}^n(S)\, \mathrm{e}^{\mathrm{i}m\eta}\, \mathrm{e}^{\mathrm{i}n\psi}, \tag{2.7}$$

where $$A_n^m = A_{n0}^m + \epsilon A_{n1}^m + \epsilon^2 \ln \epsilon A_{n12}^m + \epsilon^2 A_{n2}^m + \dots$$

and the limiting behaviour of this complete solution near the vortex core is derived for matching with the local flow. This process is required to insure that all effects of the distant parts of the vortex ring are included in the analysis. For the mean ring field ($n = 0$), the potential is already available from Lamb (1945) and from Bliss (1973), who carried the asymptotic solution for the flow near a vortex ring to $O(\epsilon^2)$.

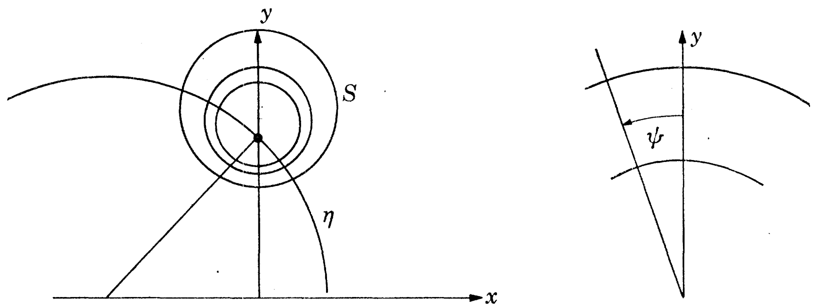

FIGURE 3. Coordinate curves S = constant and η = constant for the toroidal system: S is constant on tori coaxially located about the x axis with cross-section centres that are not coincident but approach $x = 0, y = 1$ as $S \to \infty$; η is constant on spheres centred along the x axis that pass through the point $y = 1$, $x = 0$; ψ is the azimuth angle around the torus.

For the stability problem, ϕ is constructed to represent the disturbance potential due to n waves on the filament and its limiting behaviour near the filament is then determined. This would ensure that the effects of the waves on the distant parts of the filament are included in analysis of the stability of a typical cross-section of the flow. This turns out to be unnecessary; the analysis of the behaviour of the full solution in toroidal functions shows that the effects of distant waves are *asymptotically* small. The expansion of the full solution in toroidal functions as $\epsilon \to 0$ turns out to be identical near the vortex ring to the local asymptotic solution of Laplace's equation (2.1*e*) in the coordinate system of figure 2 with no additional far-field effects. The reasons for this will be more fully discussed in § **3**, since it is not generally true that there are no far-field effects due to waves on vortex rings; it is true for the *unstable* waves because the product of their wavenumber with the core size remains finite as $\epsilon \to 0$; the waves become infinitesimally short in this limit and their far-field effects are thus asymptotically small. The detailed solution of the outer potential flow in toroidal functions and its asymptotic expansion into the coordinates of figure 2 appears in § **3**.

To obtain the linearized disturbance equations, we take the total flow field as the sum of the mean flow and the disturbance field. Using the expansion of the steady flow field in and near the core (2.2) with (2.3), the assumed form of the disturbance field (2.4) and (2.5), and the expansion of the frequency (2.6) in the governing equations (2.1) and keeping only terms linear in disturbance amplitude, we obtain for $r > 1$

$$\mathrm{i}\omega_0 u + \frac{\partial u}{\partial \theta} - 2v + \frac{\partial \pi}{\partial r} = \epsilon\left[-\left(\mathrm{i}\omega_1 + \frac{\partial U_1}{\partial r}\right)u_0 - U_1\frac{\partial u_0}{\partial r} - \frac{V_1}{r}\frac{\partial u_0}{\partial \theta} - \left(\frac{1}{r}\frac{\partial U_1}{\partial \theta} - \frac{2V_1}{r}\right)v_0\right]$$
$$+\epsilon^2\begin{bmatrix} -\left(\mathrm{i}\omega_1 + \dfrac{\partial U_1}{\partial r}\right)u_1 - U_1\dfrac{\partial u_1}{\partial r} - \dfrac{V_1}{r}\dfrac{\partial u_1}{\partial \theta} - \left(\dfrac{1}{r}\dfrac{\partial U_1}{\partial \theta} - \dfrac{2V_1}{r}\right)v_1 \\ -\left(\mathrm{i}\omega_2 + \dfrac{\partial U_2}{\partial r}\right)u_0 - U_2\dfrac{\partial u_0}{\partial r} - \dfrac{V_2}{r}\dfrac{\partial u_0}{\partial \theta} - \left(\dfrac{1}{r}\dfrac{\partial U_2}{\partial \theta} - \dfrac{2V_2}{r}\right)v_0 \end{bmatrix} + \dots \tag{2.8a}$$

$$\mathrm{i}\omega_0 v + 2u + \frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial \pi}{\partial \theta} = \epsilon\left[-\left(\mathrm{i}\omega_1 + \frac{1}{r}\frac{\partial V_1}{\partial \theta} + \frac{U_1}{r}\right) v_0 - \left(\frac{\partial V_1}{\partial r} + \frac{V_1}{r}\right) u_0 - U_1 \frac{\partial v_0}{\partial r} - \frac{V_1}{r}\frac{\partial v_0}{\partial \theta}\right]$$
$$+\epsilon^2 \begin{bmatrix} -\left(\mathrm{i}\omega_1 + \frac{1}{r}\frac{\partial V_1}{\partial \theta} + \frac{U_1}{r}\right) v_1 - \left(\frac{\partial V_1}{\partial r} + \frac{V_1}{r}\right) u_1 - U_1 \frac{\partial v_1}{\partial r} - \frac{V_1}{r}\frac{\partial v_1}{\partial \theta} \\ -\left(\mathrm{i}\omega_2 + \frac{1}{r}\frac{\partial V_2}{\partial \theta} + \frac{U_2}{r}\right) v_0 - \left(\frac{\partial V_2}{\partial r} + \frac{V_2}{r}\right) u_0 - U_2 \frac{\partial v_0}{\partial r} - \frac{V_2}{r}\frac{\partial v_0}{\partial \theta} \end{bmatrix} + \dots \tag{2.8b}$$

$$\mathrm{i}\omega_0 w + \frac{\partial w}{\partial \theta} + \mathrm{i}k\pi = \epsilon\left[-(\mathrm{i}\omega_1 + r\cos\theta)\, w_0 - \frac{V_1}{r}\frac{\partial w_0}{\partial \theta} - U_1\frac{\partial w_0}{\partial r} + \mathrm{i}kr\sin\theta\,\pi_0\right]$$
$$+\epsilon^2 \begin{bmatrix} -(\mathrm{i}\omega_1 + r\cos\theta)\, w_1 - \frac{V_1}{r}\frac{\partial w_1}{\partial \theta} - U_1\frac{\partial w_1}{\partial r} + \mathrm{i}kr\sin\theta\,\pi_1 \\ -(\mathrm{i}\omega_2 + U_1\sin\theta + V_1\cos\theta - r^2\sin\theta\cos\theta)\, w_0 \\ -\frac{V_2}{r}\frac{\partial w_0}{\partial \theta} - U_2\frac{\partial w_0}{\partial r} - \mathrm{i}kr^2\sin^2\theta\,\pi_0 \end{bmatrix} + \dots \tag{2.8c}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \mathrm{i}kw = \epsilon[-\sin\theta\, u_0 - \cos\theta\, v_0 + \mathrm{i}kr\sin\theta\, w_0]$$
$$+\epsilon^2\begin{bmatrix} -\sin\theta\, u_1 - \cos\theta\, v_1 + \mathrm{i}kr\sin\theta\, w_1 \\ +r\sin^2\theta\, u_0 + r\sin\theta\cos\theta\, v_0 - \mathrm{i}kr^2\sin^2\theta\, w_0 \end{bmatrix} + \dots \tag{2.8d}$$

for $r > 1$

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2} - k^2\phi = \epsilon\left[-\sin\theta\frac{\partial \phi_0}{\partial r} - \frac{\cos\theta}{r}\frac{\partial \phi_0}{\partial \theta} - 2k^2 r\sin\theta\,\phi_0\right]$$
$$+\epsilon^2\left[-\sin\theta\frac{\partial \phi_1}{\partial r} - \frac{\cos\theta}{r}\frac{\partial \phi_1}{\partial \theta} - 2k^2 r\sin\theta\,\phi_1 + r\sin^2\theta\frac{\partial \phi_0}{\partial r}\right.$$
$$\left. + \frac{\sin 2\theta}{2}\frac{\partial \phi_0}{\partial \theta} + 3k^2r^2\sin^2\theta\,\phi_0\right]. \tag{2.8e}$$

The various features of these equations will be discussed when the analysis is considered in more detail. The boundary conditions are as follows:

(i) as $r \to \infty$, the perturbations match the full solution for waves on a vortex ring expressed in toroidal functions.

(ii) at $r = 0$, the solution is non-singular.

(iii) at the edge of the vortex core, defined by $r = \Theta(\theta, \epsilon) + \tilde{f}(t, \theta, s; \epsilon)$, the perturbation quantities satisfy the kinematic condition

$$\mathrm{D}[r - \Theta(\theta; \epsilon) - \tilde{f}(t, \theta, s; \epsilon)]/\mathrm{D}t = 0. \tag{2.9}$$

(iv) at the edge of the core, the pressure is continuous.

After linearization in amplitude $\tilde{f}$, the kinematic boundary condition becomes

$$\frac{\partial \tilde{f}}{\partial t} + \frac{1}{\Theta^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial \tilde{f}}{\partial \theta} + \left(\frac{1}{\Theta^2}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\frac{\partial^2\Phi}{\partial\theta\,\partial r} - \frac{2}{\Theta^3}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\frac{\partial \Phi}{\partial \theta} - \frac{\partial^2\Phi}{\partial r^2}\right)\tilde{f} + \frac{1}{\Theta^2}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\frac{\partial \tilde{\phi}}{\partial \theta} - \frac{\partial \tilde{\phi}}{\partial r} = 0\bigg|_{r=\Theta(\theta;\epsilon)}, \tag{2.10a}$$

$$\frac{\partial \tilde{f}}{\partial t} + \frac{V}{\Theta}\frac{\partial \tilde{f}}{\partial \theta} + \left(\frac{1}{\Theta}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\frac{\partial V}{\partial r} - \frac{1}{\Theta^2}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}V - \frac{\partial U}{\partial r}\right)\tilde{f} + \frac{1}{\Theta}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\tilde{v} - \tilde{u} = 0\bigg|_{r=\Theta(\theta;\epsilon)} \tag{2.10b}$$

and the condition for the pressure is found to be

$$\tilde{p} = -\left(\frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r}\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{\Theta^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial^2\Phi}{\partial r\,\partial\theta} - \frac{1}{\Theta^3}\left(\frac{\partial \Phi}{\partial \theta}\right)^2\right)\tilde{f} - \frac{\partial \tilde{\phi}}{\partial t} - \frac{\partial \Phi}{\partial r}\frac{\partial \tilde{\phi}}{\partial r} - \frac{1}{\Theta^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial \tilde{\phi}}{\partial \theta}\bigg|_{r=\Theta(\theta;\epsilon)}. \tag{2.11}$$

Because of the continuity of the mean-flow pressure and velocity along the core boundary $r = \Theta(\theta)$, the terms proportional to $\tilde{f}$ (and $\partial\tilde{f}/\partial\theta$) in equation (2.10*a*) and (2.10*b*) are equal and are eliminated by subtraction in the final boundary condition for $\tilde{\phi}$ and $\tilde{u}$. The term proportional to $\tilde{f}$ in the pressure condition (2.11) is zero.

The boundary conditions at $r = 1$ are found by substituting $\Theta = 1+\alpha(\theta;\epsilon)$ and performing a Taylor series expansion of the boundary conditions (2.10) and (2.11) about $r = 1$. The kinematic boundary conditions become

$$\frac{\partial\tilde{f}}{\partial t}+\frac{\partial\tilde{f}}{\partial\theta}-\tilde{u} = \epsilon\left[-\mathrm{i}\omega_1\tilde{f}_0-V_1\frac{\partial\tilde{f}_0}{\partial\theta}+\frac{\partial U_1}{\partial r}\tilde{f}_0\right]+\epsilon^2\left[-\mathrm{i}\omega_1\tilde{f}_1-V_1\frac{\partial\tilde{f}_1}{\partial\theta}+\frac{\partial U_1}{\partial r}\tilde{f}_1 -\mathrm{i}\omega_2\tilde{f}_0-V_2\frac{\partial\tilde{f}_0}{\partial\theta}+\frac{\partial U_2}{\partial r}\tilde{f}_0+\alpha_2\frac{\partial\tilde{u}_0}{\partial r}-\frac{\mathrm{d}\alpha_2}{\mathrm{d}\theta}v_0\right], \tag{2.12a}$$

$$\tilde{u}-\frac{\partial\tilde{\phi}}{\partial r} = \epsilon^2\left[\alpha_2\left(\frac{\partial^2\tilde{\phi}_0}{\partial r^2}-\frac{\partial\tilde{u}_0}{\partial r}\right)+\frac{\mathrm{d}\alpha_2}{\mathrm{d}\theta}\left(-\frac{\partial\tilde{\phi}_0}{\partial\theta}+\tilde{v}_0\right)\right], \tag{2.12b}$$

where $\alpha(\theta;\epsilon) = \epsilon\alpha_1(\theta)+\epsilon^2\alpha_2(\theta)+\dots$.

The condition for pressure at $r = 1$ is

$$\tilde{p}+\frac{\partial\tilde{\phi}}{\partial t}+\frac{\partial\tilde{\phi}}{\partial\theta} = \epsilon\left[-\mathrm{i}\omega_1\tilde{\phi}_0-\frac{\partial\Phi_1}{\partial\theta}\frac{\partial\tilde{\phi}_0}{\partial\theta}\right]+\epsilon^2\left[-\mathrm{i}\omega_1\tilde{\phi}_1-\frac{\sin\theta}{4}\frac{\partial\tilde{\phi}_1}{\partial\theta}-\mathrm{i}\omega_2\tilde{\phi}_0-\frac{\partial\Phi_2}{\partial r}\frac{\partial\tilde{\phi}_0}{\partial r}-\frac{\partial\Phi_2}{\partial\theta}\frac{\partial\tilde{\phi}_0}{\partial\theta} +\alpha_2\left(-\frac{\partial\tilde{p}_0}{\partial r}-\mathrm{i}\omega_0\frac{\partial\tilde{\phi}_0}{\partial r}+2\frac{\partial\tilde{\phi}_0}{\partial\theta}-\frac{\partial^2\tilde{\phi}_0}{\partial\theta\,\partial r}\right)\right]. \tag{2.13}$$

Since to lowest order, the mean flow field is that for a straight filament and the effects of curvature can be seen to be absent in both the perturbation equations and the boundary conditions, it follows that the disturbance field is that for waves on a straight filament, and ω_0 is the corresponding eigenfrequency for these waves. The higher-order solutions to the linearized stability analysis determine the effects of the mean flow of the ring on the characteristics of these waves including the changes in the disturbance flow field and the frequency. Whenever ω has a negative imaginary part, the flow is unstable. Values of $\omega_0 = 0$ will be shown to be possible points of instability for the vortex ring. The analysis presented in this paper shows that $\omega_1 = 0$ and verifies our earlier speculation that ω_{12} and ω_2 have numerical values such that the thin vortex ring with *constant vorticity* is always unstable.

3. The potential flow solution for waves on a vortex ring in the limit $\epsilon\to 0$, $ka \sim O(1)$

In this section we show that there are no far-field effects from distant elements of the ring in the limit $\epsilon\to 0$ and conclude that the inner potential equation (2.8*e*) with boundary condition $\phi\to 0$ as $r\to 0$ describes the flow for $r > 1$ for the unstable waves.

To construct the potential for the flow outside the vortex core due to waves on the ring, we express the solution in the toroidal coordinates (S, η, ψ) of figure 3, chosen so that the vortex core is a surface S = constant. The relations between the coordinates S, η, the spherical coordinates x, y, and the local curved cylindrical coordinates r, θ, of figure 2 are given by

$$\epsilon r\cos\theta = x = -\frac{q\sin\eta}{S-\cos\eta}, \quad 1+\epsilon r\sin\theta = y = q\frac{(S^2-1)^{\frac{1}{2}}}{S-\cos\eta}, \tag{3.1}$$

where q is the scale factor of the toroidal system (see Bateman, 1955).

If q is chosen as $(1-\epsilon^2)^{\frac{1}{2}}$, then the core boundary is the coordinate surface $S = S_0 = 1/\epsilon$.

The general form of the solution for a disturbance of n waves around the azimuth is

$$\phi(S, \eta, \psi) = (S-\cos\eta)^{\frac{1}{2}} \sum_m \sum_n A_n^m P_{m-\frac{1}{2}}^n(S)\, e^{im\eta}\, e^{in\psi}, \tag{2.7}$$

where $P_{m-\frac{1}{2}}^n(S)$ is the associated Legendre function. A Legendre function of the second kind $Q_{m-\frac{1}{2}}^n$ is not allowed since it is singular along the x axis.

The coefficients A_n^m are determined by satisfying the boundary conditions on the edge of the core. In general, since $\eta \neq \theta$, it requires an infinite sum over the functions $e^{im\eta}$ to represent the bending wave $e^{\pm i\theta}$. However, in the limit $\epsilon \to 0$, it is possible to obtain an asymptotic expression for the solution (2.7) which requires only a few terms since, in the limit $\epsilon \to 0$, the relation (3.1) between r, θ, and S, η can be expanded as

$$S \simeq \frac{1}{\epsilon r} - \frac{1}{2}\left(\frac{1}{r^2}-1\right)\sin\theta + \epsilon\left[\frac{-1}{8r^3}+\frac{3r}{8}-\frac{1}{4r}+\left(\frac{3}{8r^3}-\frac{r}{8}-\frac{1}{4r}\right)\sin^2\theta\right] + O(\epsilon^2) \tag{3.2}$$

and

$$\sin\eta \simeq -\cos\theta + \frac{\epsilon}{4}\left(r+\frac{1}{r}\right)\sin 2\theta - \epsilon^2\left[\left(\frac{3}{8}\left(r+\frac{1}{r}\right)^2-\frac{1}{2}\right)\sin^2\theta\cos\theta - \frac{1}{8}\left(r+\frac{1}{r}\right)^2\cos\theta\right] + O(\epsilon^3), \tag{3.3}$$

$$\cos\eta \simeq \sin\theta + \frac{\epsilon}{4}\left(r+\frac{1}{r}\right)(1+\cos 2\theta) + \epsilon^2\left[\left(\frac{-3}{8}\left(r+\frac{1}{r}\right)^2+\frac{1}{2}\right)\sin\theta\cos^2\theta\right] + O(\epsilon^3). \tag{3.4}$$

From (3.2)–(3.4), we can see that to lowest order, $S \sim 1/\epsilon r$, $\sin\eta \sim -\cos\theta$ and $\cos\eta \sim \sin\theta$ so that only $e^{i\eta}$ would be required to represent the mode $e^{i\theta}$; terms of the form $e^{\pm i2\eta}$, etc., appear at higher orders.

To obtain the expansion of (2.7) into r, θ coordinates, we also expand the associated Legendre function $P_{m-\frac{1}{2}}^n(S)$.

The integral representation of this function is

$$P_{m-\frac{1}{2}}^n(S) \propto (S^2-1)^{n/2}\int_0^\infty (S+\cosh t)^{-m-n-\frac{1}{2}}(\sinh t)^{2m}\,dt \tag{3.5}$$

while the integral representation for the Bessel functions, which are the solutions for the related problem of waves on a straight filament, is

$$K_m(x) \propto x^m \int_0^\infty e^{-x\cosh t}(\sinh t)^{2m}\,dt. \tag{3.6}$$

The proper expansion of (3.5) is obtained for $\epsilon \to 0$ with $k \sim O(1)$. (The wave number has been non-dimensionalized by core size a.) With this non-dimensionalization $k/\epsilon = n$, the number of waves on the ring. In order to hold $k \sim O(1)$ as $\epsilon \to 0$, we require $n = k/\epsilon$ as $\epsilon \to 0$, in other words, the simultaneous limit of decreasing core size and increasing wavenumber.

If we set $S = 1/\epsilon r$ and $n = k/\epsilon$, in (3.5) and take the limit $\epsilon \to 0$, we obtain (to lowest order),

$$P_{m-\frac{1}{2}}^n(S) \propto r^{m+\frac{1}{2}}\int_0^\infty (1+\epsilon r\cosh t)^{-k/\epsilon}(\sinh t)^{2m}\,dt.$$

Since $\lim_{\epsilon\to 0}(1+\epsilon r\cosh t)^{-k/\epsilon} \simeq e^{-kr\cosh t}$, $P_{n-\frac{1}{2}}^m(S)$ is related to $K_m(kr)$ as

$$\lim_{\epsilon\to 0} P_{m-\frac{1}{2}}^n(S) \propto r^{m+\frac{1}{2}}\int_0^\infty e^{-kr\cosh t}(\sinh t)^{2m}\,dt \propto \sqrt{r}\,K_m(kr). \tag{3.7}$$

To obtain an expression to $O(\epsilon^2)$, considerable additional manipulation is required. The final result in this limit is (Tsai 1976)

$$P^n_{m-\frac{1}{2}}(S) \propto \left(\frac{z}{n}\right)^{m+\frac{1}{2}}\left[\frac{K_m}{z^m}+\left(\frac{z}{n}\right)E_m+\left(\frac{z}{n}\right)^2 F_m+0\left(\left(\frac{z}{n}\right)^3\right)\right], \tag{3.8}$$

where $z = kr'$,

$$E_m = \frac{z}{2}\frac{d^2}{dz^2}\left(\frac{K_m}{z^m}\right)+(m+\tfrac{1}{2})\frac{d}{dz}\left(\frac{K_m}{z^m}\right)-\frac{K_m}{2z^{m-1}},$$

$$F_m = \frac{z^2}{2}\frac{d^4}{dz^4}\left(\frac{K_m}{z^m}\right)+\left(\frac{m}{2}+\frac{7}{12}\right)z\frac{d^3}{dz^3}\left(\frac{K_m}{z^m}\right)+\left(\frac{-z^2}{4}+\tfrac{1}{2}(m+\tfrac{1}{2})(m+\tfrac{3}{2})\right)\frac{d^2}{dz^2}\left(\frac{K_m}{z^m}\right)$$
$$+\frac{k}{2r'}-\frac{z}{2}(m+\tfrac{3}{2})\frac{d}{dz}\left(\frac{K_m}{z^m}\right)+\left(\frac{z^2}{8}-\tfrac{1}{2}(m+\tfrac{1}{2})\left(\frac{1}{r'^2}-1\right)\right)\frac{K_m}{z^m},$$

with $$r' = r+\epsilon[\tfrac{1}{2}(1-r^2)\sin\theta]+\epsilon^2\left[\frac{1}{16r}-\frac{3r}{8}+\frac{5}{16}r^3+\frac{1}{16}\left(\frac{1}{r}+2r-3r^3\right)\cos 2\theta\right]+O(\epsilon^3)$$

and K_m are modified Bessel functions with argument z.

With equations (3.2), (3.3), (3.4) and (3.8) it is tedious but straightforward to express (2.7) as a series solution in r, θ, s involving Bessel functions $K_m(kr)$. The most significant result of this analysis (Tsai 1976) is that to $O(\epsilon^2)$ the potential flow solution so obtained could have been obtained directly from the solution of the potential flow equations in inner variables (equation (2.1 *e*)); i.e. there are no additional far-field effects (these would have shown up as Bessel functions $I_m(kr)$ since far-field effects have to satisfy the homogeneous equation). To verify this, a similar analysis was applied to the stability of the vortex pair for asymptotically short waves ($k \sim 0(1)$ as $\epsilon \to 0$). It is easy to see in this case that the far-field effects decay as $e^{-kr/\epsilon}$.

Therefore, we shall not deal any further with the solution in toroidal coordinates but shall return to the solution of the inner problem which contains all of the stability information.

4. The mean flow field of the vortex ring near the core

To begin the stability analysis, we must have available the mean flow field of the vortex ring to $O(\epsilon^2)$. Although various features of this flow have been described by Fraenkel (1972) and Bliss (1973), the complete velocity and pressure field have not been given to $O(\epsilon^2)$ in a form suitable for our analysis. The solution for this flow is presented in this section. This solution is equivalent to that given by Fraenkel (1972). His analysis constructed the stream function for constant ζ/r with the core boundary determined by an asymptotic mapping technique for small cores. Our solution will be directly obtained by a perturbation expansion in ϵ of the equations of motion and boundary conditions. The reader may wish to skip directly to the results presented in equations (4.22), (4.23), before going on to the stability analysis of § 5. However, understanding the properties of the mean flow is crucial to understanding the instability of the ring.

For a complete solution of the steady flow field of the vortex ring, the self-induced propagation velocity could be expanded in an asymptotic series in ϵ and determined along with the flow field. However, from the work of Fraenkel (1972), the propagation velocity is already known to $O(\epsilon^2)$ so that it will *not* be taken as an unknown in our analysis. To $O(\epsilon^2)$ the non-dimensional, self-induced propagation velocity of the vortex ring is

$$V_0 = \frac{\epsilon}{2}\left(\ln\frac{8}{\epsilon}-\frac{1}{4}\right). \tag{4.1}$$

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The problem is formulated in a coordinate system fixed in the ring so that the mean flow is steady. For this steady flow, the boundary conditions are obtained as follows:

(i) as $r \to \infty$, the solution matches asymptotically with the inner limit of the outer solution for a thin vortex ring (Bliss 1973);

$$\lim_{r\to\infty} U(r,\theta) = \epsilon[(-\tfrac{3}{8}-\tfrac{1}{2}\ln r)\cos\theta] + \epsilon^2\left[r\left(-\tfrac{3}{8}\ln\frac{8}{\epsilon}+\tfrac{3}{8}\ln r+\tfrac{1}{2}\right)\sin 2\theta\right]$$

and
$$\lim_{r\to\infty} V(r,\theta) = \frac{1}{r} + \epsilon[(-\tfrac{1}{8}+\tfrac{1}{2}\ln r)\sin\theta] + \epsilon^2\left[r\left(-\tfrac{3}{8}\ln\frac{8}{\epsilon}+\tfrac{3}{8}\ln r+\tfrac{5}{16}\right)\cos 2\theta\right];$$

(ii) as $r \to 0$, disturbances are required to be finite;

(iii) the kinematic condition at the core boundary $r = \Theta(\theta)$ expanded about $r = 1$ (2.12) relates the radial velocity U to the shape of the core;

(iv) the dynamic condition at the core boundary $r = \Theta(\theta)$ expanded about $r = 1$ (2.13) relates the pressure and velocity potential to the shape of the core.

When the assumed form of the velocity/pressure/potential/shape-change solution (2.2) is substituted into the governing equation (2.1) and the boundary conditions (2.12) and (2.13), we obtain the set of governing equations and boundary conditions for the various terms in (2.2).

To lowest order, the flow is just that of a straight line filament. The leading term in the expansion is a filament with constant vorticity without axial velocity; this solution is

$$\{U_0, V_0, W_0, P_0, \Phi_0, \Theta_0\} = \{0, r, 0, \tfrac{1}{2}r^2-1, \theta, 1\}. \tag{4.2}$$

The perturbation solution automatically incorporates the stretching of vortex lines to produce a ring of ζ/r = constant at $O(\epsilon)$. At the next order, the governing equations are for $r < 1$ (from 2.1 a, b, d)

$$\left.\begin{aligned} \frac{\partial U_1}{\partial\theta} - 2V_1 &= -\frac{\partial P_1}{\partial r}, \\ 2U_1 + \frac{\partial V_1}{\partial\theta} &= -\frac{1}{r}\frac{\partial P_1}{\partial\theta}, \\ \frac{\partial U_1}{\partial r} + \frac{U_1}{r} + \frac{1}{r}\frac{\partial V_1}{\partial\theta} &= -r\cos\theta. \end{aligned}\right\} \tag{4.3}$$

Introducing the stream function ψ_1 such that

$$\left.\begin{aligned} U_1 &= -\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}, \\ V_1 &= \frac{\partial\psi_1}{\partial r} - r^2\sin\theta \end{aligned}\right\} \tag{4.4}$$

and eliminating the pressure from (4.3), we obtain for $r < 1$

$$\frac{\partial^2\psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2} = 5r\sin\theta \tag{4.5}$$

and from (2.1 e) for $r > 1$

$$\frac{\partial^2\Phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi_1}{\partial\theta^2} = -\frac{\cos\theta}{r}. \tag{4.6}$$

With boundary conditions at $r = 1$

$$\left.\begin{aligned} U_1 - \frac{\mathrm{d}\alpha_1}{\mathrm{d}\theta} V_0 &= 0, \\ \frac{\partial \Phi_1}{\partial r} - \frac{\mathrm{d}\alpha_1}{\mathrm{d}\theta} &= 0, \\ P_1 + \alpha_1 \frac{\partial P_0}{\partial r} &= \alpha_1 - \frac{\partial \Phi_1}{\partial \theta} \end{aligned}\right\} \tag{4.7}$$

and as $r \to \infty$

$$\Phi_1 \to -\tfrac{1}{2} r (\tfrac{1}{4} + \ln r) \cos\theta. \tag{4.8}$$

The solution to this problem was given by Bliss (1970) as

$$\{U_1, V_1, W_1, P_1, \Phi_1, \alpha_1(0)\} = \{\tfrac{5}{8}(1-r^2)\cos\theta,\ (-\tfrac{5}{8}+\tfrac{7}{8}r^2)\sin\theta,\ 0,\ (-\tfrac{5}{8}r+\tfrac{3}{8}r^3)\sin\theta,\ (\tfrac{1}{8}r-\tfrac{3}{8}r^{-1}-\tfrac{1}{2}r\ln r)\cos\theta,\ 0\}. \tag{4.9}$$

This solution contains the lowest-order effects of ring curvature. Note that all flow quantities depend upon $\sin\theta$ or $\cos\theta$; physically, these terms represent the stretching of vorticity and the effects of conservation of mass as the vortex elements convect around the curved core axis. The problem defined by (4.3) → (4.8) also allows an eigensolution which represents a uniform displacement of the core boundary in the direction of propagation. However, since we are free to centre the vortex with the coordinate system, this eigensolution may be omitted.

The solution to $O(\epsilon^2)$ is governed by the following equations for $r < 1$:

$$\frac{\partial U_2}{\partial \theta} - 2V_2 + \frac{\partial P_2}{\partial r} = -U_1 \frac{\partial U_1}{\partial r} - \frac{V_1}{r}\frac{\partial U_1}{\partial \theta} + \frac{V_1^2}{r}, \tag{4.10a}$$

$$2U_2 + \frac{\partial V_2}{\partial \theta} + \frac{1}{r}\frac{\partial P_2}{\partial \theta} = -U_1\frac{\partial V_1}{\partial r} - \frac{V_1}{r}\frac{\partial V_1}{\partial \theta} - \frac{U_1 V_1}{r}, \tag{4.10b}$$

$$\frac{\partial U_2}{\partial r} + \frac{U_2}{r} + \frac{1}{r}\frac{\partial V_2}{\partial \theta} = -U_1\sin\theta + r^2\sin\theta\cos\theta - V_1\cos\theta. \tag{4.10c}$$

When the first-order solution (4.9) is substituted into the continuity equation, (4.10c) becomes

$$\frac{\partial U_2}{\partial r} + \frac{U_2}{r} + \frac{1}{r}\frac{\partial V_2}{\partial \theta} = \tfrac{3}{8} r^2 \sin 2\theta. \tag{4.11}$$

A stream function ψ_2 is then introduced of the following form

$$U_2 = -\frac{1}{r}\frac{\partial \psi_2}{\partial \theta}, \qquad V_2 = \frac{\partial \psi_2}{\partial \theta} - \tfrac{3}{16} r^3 \cos 2\theta \tag{4.12}$$

so that (4.11) is identically satisfied. From (4.10a, b) and (4.12), we obtain a single equation for the stream function ψ_2

$$\frac{\partial^2 \psi_2}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi_2}{\partial \theta^2} = \tfrac{3}{4} r^2 \cos 2\theta. \tag{4.13}$$

For $r > 1$, the governing equation is [from (2.8*e*)]

$$\frac{\partial^2 \Phi_2}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi_2}{\partial \theta^2} = -\sin\theta\frac{\partial \Phi_1}{\partial r}-\frac{\cos\theta}{r}\frac{\partial \Phi_1}{\partial \theta}+r\sin^2\theta\frac{\partial \Phi_0}{\partial r}+\frac{\sin 2\theta}{2}\frac{\partial \Phi_0}{\partial \theta}. \tag{4.14}$$

Substituting ϕ_1 from (4.9), we obtain

$$\frac{\partial^2 \Phi_2}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi_2}{\partial \theta^2} = (-\tfrac{3}{8}r^{-2}+\tfrac{3}{4})\sin 2\theta. \tag{4.15}$$

The boundary conditions are –

$$\left.\begin{aligned} &\text{at } r=1: && U_2-\frac{\mathrm{d}\alpha_2}{\mathrm{d}\theta}=0,\\ & && \frac{\partial \Phi_2}{\partial r}-\frac{\mathrm{d}\alpha_2}{\mathrm{d}\theta}=0,\\ & && P_2+\frac{\partial \Phi_2}{\partial \theta}=-\frac{1}{2}\left(\frac{\partial \Phi_1}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial \Phi_1}{\partial \theta}\right)^2;\end{aligned}\right\} \tag{4.16}$$

$$\text{at } r\to\infty: \quad \Phi_2 \to r^2\left[-\tfrac{3}{16}\ln\frac{8}{\epsilon}+\tfrac{3}{16}\ln r+\tfrac{5}{32}\right]\sin 2\theta.$$

We assume a solution to (4.13) and (4.15) in the form

$$\Psi_2 = \sum_{m=-\infty}^{\infty} A_m r^m \mathrm{e}^{\mathrm{i}m\theta}+\Psi_{2p}(r)\sin 2\theta \tag{4.17}$$

and

$$\Phi_2 = \sum_{m=-\infty}^{\infty} B_m r^m \mathrm{e}^{\mathrm{i}m\theta}+\Phi_{2p}(r)\sin 2\theta. \tag{4.18}$$

From the governing equations (4.13), (4.15) and the boundary conditions (4.16), we determine that

$$\left.\begin{aligned} A_m &= 0 \quad \text{for all } m,\\ B_m &= 0 \quad m \neq 2,\end{aligned}\right\} \tag{4.19}$$

and that

$$\Phi_2 = \left[\tfrac{5}{32}r^2-\tfrac{9}{32}r^{-2}+\tfrac{3}{32}+\tfrac{3}{16}r^2\ln r-\tfrac{3}{16}\ln\frac{8}{\epsilon}(r^2-r^{-2})\right]\sin 2\theta \tag{4.20}$$

and

$$\Psi_2 = \left[\left(\tfrac{15}{32}-\tfrac{3}{8}\ln\frac{8}{\epsilon}\right)r^2+\frac{r^4}{16}\right]\cos 2\theta. \tag{4.21}$$

To this order, the mean flow field near the vortex core includes a straining flow ($r^2 \sin 2\theta$, etc.) although, due to the presence of the logarithmic terms, this is not strictly a local two-dimensional strain (or stagnation-point flow) but rather its analogue for small filament curvature.

We have now completed the analysis of the steady flow field in and near the curved vortex filament (of constant vorticity) to $O(\epsilon^2)$. The complete solution is as follows –

for $r < 1$:

$$U = \epsilon\tfrac{5}{8}(1-r^2)\cos\theta+\epsilon^2\left[\left(\tfrac{15}{16}-\tfrac{3}{4}\ln\frac{8}{\epsilon}\right)r+\frac{r^3}{8}\right]\sin 2\theta+\ldots,$$

$$V = r+\epsilon(-\tfrac{5}{8}+\tfrac{7}{8}r^2)\sin\theta+\epsilon^2\left[\left(\tfrac{15}{16}-\tfrac{3}{4}\ln\frac{8}{\epsilon}\right)r+\frac{r^3}{16}\right]\cos 2\theta+\ldots,$$

$$P = \frac{r^2}{2}-1+\epsilon(-\tfrac{5}{8}r+\tfrac{3}{8}r^3)\sin\theta+\epsilon^2[\tfrac{5}{32}r^2-\tfrac{9}{128}r^4-\tfrac{3}{32}+(\tfrac{15}{64}r^2-\tfrac{5}{32}r^4)\cos 2\theta];$$

for $r > 1$:

$$\Phi = \theta + \epsilon(\tfrac{1}{8}r - \tfrac{3}{8}r^{-1} - \tfrac{1}{2}r\ln r)\cos\theta + \epsilon^2\left[\tfrac{5}{32}r^2 - \tfrac{9}{32}r^{-2} + \tfrac{3}{32} + \tfrac{3}{16}r^2\ln r - \tfrac{3}{16}\ln\frac{8}{\epsilon}(r^2 - r^{-2})\right]\sin 2\theta + \dots. \quad (4.22)$$

The shape of the core boundary is slightly elliptical with

$$\Theta = 1 + \epsilon^2\left(-\tfrac{17}{32} + \tfrac{3}{8}\ln\frac{8}{\epsilon}\right)\cos 2\theta. \quad (4.23)$$

This expression agrees with the result of Fraenkel (1972) for the shape of the vortex core found by a very different method and serves as a check on the structure of the mean flow presented in (4.22).

The form of the mean flow field of the vortex ring reflects clearly the development of the solution as an asymptotic expansion in the curvature parameter $\epsilon = a/R$. The zero-order solution is that for a straight filament; to $O(\epsilon)$, the curvature introduces $\cos\theta$, $\sin\theta$ terms representing the stretching of vorticity and the speeding and slowing of the local flow around the curved centreline; the $O(\epsilon^2)$ terms introduce a local quasi-two-dimensional straining or stagnation-point flow ($\sin 2\theta$, $\cos 2\theta$). To order (ϵ^2), the vortex core becomes slightly elliptical as does a line filament in a weak straining flow (Moore & Saffman 1971). The straining flow is responsible for the instability of bending waves on the vortex ring. Unfortunately, in order to obtain this instability, it is necessary to carry the stability analysis to $O(\epsilon^2)$ to include the effects of both curvature and strain on the waves.

5. An overview of the stability analysis

We now outline the calculation of the stability of waves on the vortex ring. Since we are interested primarily in bending waves, only this mode of disturbance will be considered although the stability of other waves such as bulge waves or waves which distort the shape of the vortex cross-section could be considered by the same technique. In fact, because of the form of the perturbation equations (2.8) and the mean flow solution (4.22), bulge modes and shape-change modes appear to higher order, as will be seen.

In such a complex problem, it may be useful to have an overview of the major features of the problem. Such an overview is given in this section, using an extended operator notation that we shall refer to as a procedure. The actual details of the stability analysis appear in §§ 6–8.

From the results of § 4 (4.22), the mean flow can be written in the form

$$\boldsymbol{Q} = \boldsymbol{Q}_0(r) + \epsilon(\boldsymbol{Q}_1(r)\,e^{i\theta} + \boldsymbol{Q}_1(r)\,e^{-i\theta}) + \epsilon^2(\boldsymbol{Q}_2(r)\,e^{2i\theta} + \overline{\boldsymbol{Q}}_2(r)\,e^{-2i\theta}) + \dots, \quad (5.1)$$

where a bar denotes complex conjugate and where terms of $O(\epsilon^2\ln\epsilon)$ have been incorporated as $O(\epsilon^2)$. In (5.1), $\boldsymbol{Q}$ is the extended mean flow field vector

$$\boldsymbol{Q} = \{U, V, W, P, \Phi, \Theta\}. \quad (5.2)$$

The disturbance flow is then expanded (as in (2.4)) as

$$\tilde{\boldsymbol{q}} = \boldsymbol{q}\,e^{i(\omega t + ks)} = \{\boldsymbol{q}_0 + \epsilon\boldsymbol{q}_1 + \epsilon^2\boldsymbol{q}_2 + \dots\}\,e^{i(\omega t + ks)}, \quad (5.3)$$

where $\boldsymbol{q}$ is the extended perturbation vector $\boldsymbol{q} = \{u, v, w, p, \phi, f\}$ and s is the arc length variable scaled by ϵ. Of course, we can only admit an integer number of waves on the ring, but for now k

will be taken as a continuous variable. For a given k, we can always find a core size a such that an integer number of waves of wavenumber k can fit on the ring.

The frequency is also expanded as

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots. \tag{5.4}$$

If the lowest-order solution $\boldsymbol{q}_0$ is assumed to be of the form

$$\boldsymbol{q}_0 = \boldsymbol{q}_0^0(r) + \boldsymbol{q}_0^1\,\mathrm{e}^{\mathrm{i}\theta} + \overline{\boldsymbol{q}}_0^1\,\mathrm{e}^{-\mathrm{i}\theta} + \boldsymbol{q}_0^2\,\mathrm{e}^{2\mathrm{i}\theta} + \overline{\boldsymbol{q}}_0^2\,\mathrm{e}^{-2\mathrm{i}\theta} + \boldsymbol{q}_0^3\,\mathrm{e}^{3\mathrm{i}\theta} + {}^{3}_{0}\,\mathrm{e}^{-3\mathrm{i}\theta} + \dots \tag{5.5}$$

in the governing equations and boundary conditions, a set of homogeneous eigenvalue problems is obtained for all of the various waves on a straight filament of constant vorticity. We shall denote these problems by

$$\left.\begin{aligned} &\mathscr{L}_0^0(\omega_0^0;k)\,\boldsymbol{q}_0^0 = 0; \\ &\mathscr{L}_0^1(\omega_0^1;k)\,\boldsymbol{q}_0^1 = 0; \quad \overline{\mathscr{L}}_0^1(\overline{\omega}_0^1;k)\,\overline{\boldsymbol{q}}_0^1 = 0; \\ &\mathscr{L}_0^2(\omega_0^2;k)\,\boldsymbol{q}_0^2 = 0; \quad \overline{\mathscr{L}}_0^2(\overline{\omega}_0^2;k)\,\overline{\boldsymbol{q}}_0^2 = 0; \\ &\mathscr{L}_0^3(\omega_0^3;k)\,\boldsymbol{q}_0^3 = 0; \quad \overline{\mathscr{L}}_0^3(\overline{\omega}_0^3;k)\,\overline{\boldsymbol{q}}_0^3 = 0; \end{aligned}\right\} \tag{5.6}$$

where $\mathscr{L}_0^m(\omega_0^m;k)$ is the set of linear operations (the procedure) that generates the homogeneous solution to the eigenvalue problem of waves of type m (bulge waves, $m = 0$; bending waves, $m = 1$; shape-change waves, $m \geqslant 2$) on a straight filament including solving the governing linear homogeneous differential equation both inside and outside the core and the associated linear homogeneous operation of enforcing the boundary conditions at $r = 1$. The barred quantities simply refer to the equivalent operations for the complex conjugate modes $\mathrm{e}^{-\mathrm{i}m\theta}$. The procedure $\mathscr{L}_0^m$ leads to the determination of the eigenfrequency ω_0^m at the wavenumber k for each type of wave but does not determine the wave amplitude. For a vortex without axial flow, the dispersion relation for the waves $\mathrm{e}^{-\mathrm{i}m\theta}$ is the reflexion of the dispersion relation for the waves $\mathrm{e}^{\mathrm{i}m\theta}$ (Moore & Saffman 1975; Tsai & Widnall 1976) so that $\overline{\omega}_0^m(k) = -\omega_0^m(k)$ for a given wave. (See, for example, the dispersion relation for bending waves on a straight filament shown in figure 4.) In general, for a given wavenumber k and radial mode number, the eigenvalues ω_0^m and $\overline{\omega}_0^m$ for the different types of waves are not equal. To study the stability of bending waves on the vortex ring, we pick a wave of wavenumber k and corresponding eigenfrequency ω_0^1 and analyse the corrections to the frequency due to the curvature and strain terms in the steady flow field of the ring. In general, only one bending wave can exist at a given value of ω_0 and k. However, an important exception occurs if ω_0 is chosen at a crossing point $\overline{\omega}_0^1 = \omega_0^1$ of the dispersion curves for the $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$ modes, at which both waves can exist simultaneously at arbitrary amplitudes. Note that this generally can occur only for different radial mode numbers for the $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$ modes, except when $\overline{\omega}_0^1 = \omega_0^1 = 0$. The condition $\overline{\omega}_0^1 = \omega_0^1$ is a necessary condition for the instability of a line vortex in the presence of strain (Tsai & Widnall 1976) and, as we shall see, also leads to instability in the cases we have examined for the vortex ring. In general, these particular ω_0, k combinations are not simultaneously eigenvalues for the other possible waves on the vortex filament (bulge waves and shape-change waves) and, although there may be vorticity distributions for which this is a possibility, we shall not consider this case further. These ω_0, k crossing points can be seen in figure 4.

For bending waves, we choose ω_0 as ω_0^1, dropping the identifying superscript. We shall examine in detail only those points for which $\overline{\omega}_0^1 = \omega_0^1$, although the more general case can easily be examined by setting $\overline{\boldsymbol{q}}_0^1 \equiv 0$ (if $\overline{\omega}_0^1 \neq \omega_0^1$, then $\overline{\boldsymbol{q}}_0^1 \equiv 0$). This more general case does not lead to instability.

With this particular choice of ω_0, the zero-order disturbance flow is that of the two bending waves $\boldsymbol{q}_0^1\,\mathrm{e}^{\mathrm{i}\theta}$ and $\overline{\boldsymbol{q}}_0^1\,\mathrm{e}^{-\mathrm{i}\theta}$ that can exist simultaneously at arbitrary amplitudes on the line filament.

Because of the form of the mean flow and the lowest-order solution, to obtain the solution to $O(\epsilon)$ we assume $\boldsymbol{q}_1$ in the form

$$\boldsymbol{q}_1 = \boldsymbol{q}_1^0(r) + \boldsymbol{q}_1^1(r)\,\mathrm{e}^{\mathrm{i}\theta} + \overline{\boldsymbol{q}}_1^1(r)\,\mathrm{e}^{-\mathrm{i}\theta} + \boldsymbol{q}_1^2(r)\,\mathrm{e}^{2\mathrm{i}\theta} + \overline{\boldsymbol{q}}_1^2(r)\,\mathrm{e}^{-2\mathrm{i}\theta} + \boldsymbol{q}_1^3\,\mathrm{e}^{3\mathrm{i}\theta} + \overline{\boldsymbol{q}}_1^3\,\mathrm{e}^{-3\mathrm{i}\theta} + \ldots. \tag{5.7}$$

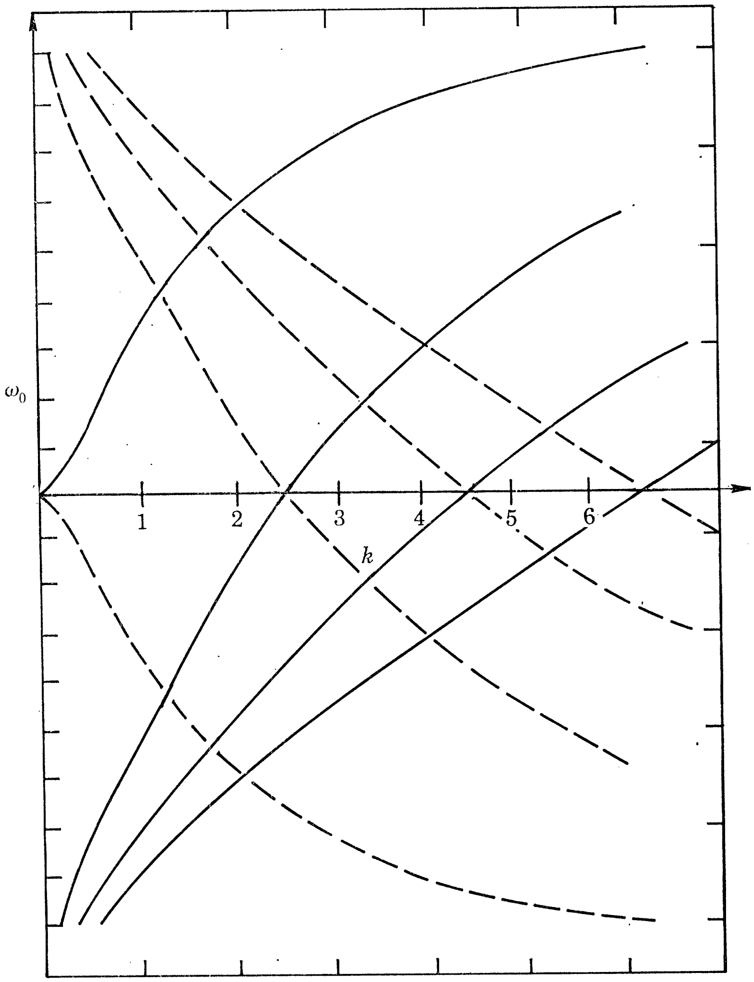

FIGURE 4. Curves of the dispersion relation for waves on a straight vortex filament with constant vorticity. ——, $\mathrm{e}^{\mathrm{i}\theta}$ mode; – – –, $\mathrm{e}^{-\mathrm{i}\theta}$ mode.

This form of solution with the governing linear equations and the associated linear boundary conditions gives rise to a set of problems which differ in several crucial respects from those of (5.6). These problems are denoted as follows:

$$\left.\begin{aligned}
&\mathscr{L}_0^0(\omega_0)\,\boldsymbol{q}_1^0 = N_1^0(\boldsymbol{q}_0, \overline{\boldsymbol{q}}_0),\\
&\mathscr{L}_0^1(\omega_0)\,\boldsymbol{q}_1^1 = -\mathrm{i}\omega_1\,N_1^1(\boldsymbol{q}_0); \quad \overline{\mathscr{L}}_0^1(\omega_0)\,\overline{\boldsymbol{q}}_1^1 = -\mathrm{i}\omega_1\,\overline{N}_1^1(\overline{\boldsymbol{q}}_0);\\
&\mathscr{L}_0^2(\omega_0)\,\boldsymbol{q}_1^2 = N_1^2(\boldsymbol{q}_0, \overline{\boldsymbol{q}}_0); \quad \overline{\mathscr{L}}_0^2(\omega_0)\,\overline{\boldsymbol{q}}_1^2 = \overline{N}_1^2(\overline{\boldsymbol{q}}_0, \boldsymbol{q}_0);\\
&\mathscr{L}_0^3(\omega_0)\,\boldsymbol{q}_1^3 = N_1^3(\boldsymbol{q}_0, \overline{\boldsymbol{q}}_0); \quad \overline{\mathscr{L}}_0^3(\omega_0)\,\overline{\boldsymbol{q}}_1^3 = \overline{N}_1^3(\overline{\boldsymbol{q}}_0, \boldsymbol{q}_0);
\end{aligned}\right\} \tag{5.8}$$

where the procedure $\mathscr{L}_0^m(\omega)$ has the same meaning as in (5.6). The notation

$$\mathscr{L}_0^m(\omega_0)\,\boldsymbol{q}_1^m = N_1^m(\boldsymbol{q}_0)$$

is the modification of those operations when both the differential equation and the boundary condition contain non-homogeneous terms with the indicated dependence on $\boldsymbol{q}_0^1$, $\overline{\boldsymbol{q}}_0^1$ and ω_1. The N's are linear in $\boldsymbol{q}_0^1$ and $\overline{\boldsymbol{q}}_0^1$.

The major differences between (5.8) and (5.6) are the presence of these non-homogeneous terms and the fact that the 'eigenvalue' ω_0 that appears in the operations indicated by $\mathscr{L}_0^m(\omega_0)$ is no longer unknown but is the ω_0 for *bending* waves on a straight filament.

The implications of these differences for the various waves in $\boldsymbol{q}_1$ are as follows:

(1) The bulge waves $\boldsymbol{q}_1^0$ and the shape-change waves $\boldsymbol{q}_1^2$ and $\overline{\boldsymbol{q}}_1^2$ are no longer governed by an eigenvalue problem, since, in general, $\omega_0 \neq \omega_0^0$ or ω_0^2, the eigenvalues for these waves. The amplitude and form of these waves is then completely determined in proportion to the amplitude of the lowest-order bending waves $\boldsymbol{q}_0^1$ and $\overline{\boldsymbol{q}}_0^1$.

(2) For the bending waves $\boldsymbol{q}_1^1$, and $\overline{\boldsymbol{q}}_1^1$, we have a forced eigenvalue problem, since

$$\omega_0 = \omega_0^1 = \overline{\omega}_0^1.$$

Because $\boldsymbol{q}_0^1$ and $\overline{\boldsymbol{q}}_0^1$ are solutions to the homogeneous problem, we must enforce a solvability condition on ω_1 to remove secular behaviour. The correction to the frequency ω_1 multiplies every term in the forcing functions $N_1^1(\boldsymbol{q}_0)$ and $\overline{N}_1^1(\overline{\boldsymbol{q}}_0)$; the only way to ensure solvability is to require $\omega_1 = 0$. This gives $\boldsymbol{q}_1^1$ and $\overline{\boldsymbol{q}}_1^1$ as the eigenmodes for bending waves on a straight filament.

(3) The higher shape change modes, $\mathrm{e}^{\pm \mathrm{i} m\theta}$, $m \geqslant 3$, are governed by a homogeneous problem with the eigenvalue already determined as ω_0^1. Since this is, in general, not an eigenvalue for these problems, the amplitude of these waves is zero to this order.

Proceeding on to $O(\epsilon^2)$, we are interested only in determining ω_2, the modification to the frequency. In this problem, ω_2 appears only in the governing equations and boundary conditions for the bending waves $\boldsymbol{q}_2^1\,\mathrm{e}^{\mathrm{i}\theta}$ and $\overline{\boldsymbol{q}}_2^1\,\mathrm{e}^{-\mathrm{i}\theta}$. The amplitude and form of $\boldsymbol{q}_2^1$ and $\overline{\boldsymbol{q}}_2^1$ are determined by the solution to the problems

$$\mathscr{L}_0^1(\omega_0)\,\boldsymbol{q}_2^1 = G_2(\omega_2, \boldsymbol{q}_0^1, \overline{\boldsymbol{q}}_0^1, \boldsymbol{q}_1^0, \boldsymbol{q}_1^2, \overline{\boldsymbol{q}}_1^2) = N_2(\omega_2, \boldsymbol{q}_0^1, \overline{\boldsymbol{q}}_0^1)$$

and

$$\mathscr{L}_0^1(\omega_0)\,\overline{\boldsymbol{q}}_2^1 = \overline{G}_2(\omega_2, \boldsymbol{q}_0^1, \overline{\boldsymbol{q}}_0^1, \boldsymbol{q}_1^0, \boldsymbol{q}_1^2, \overline{\boldsymbol{q}}_1^2) = \overline{N}_2(\omega_2, \boldsymbol{q}_0^1, \overline{\boldsymbol{q}}_0^1).$$

That is, $\boldsymbol{q}_2^1$ and $\overline{\boldsymbol{q}}_2^1$ are determined by the solution to a forced eigenvalue problem where the forcing term contains the lower-order wave solutions and the frequency correction ω_2. Since $\boldsymbol{q}_1^0$ and $\boldsymbol{q}_1^2$, $\overline{\boldsymbol{q}}_1^2$ have already been determined as being linearly proportional to $\boldsymbol{q}_0^1$ and $\overline{\boldsymbol{q}}_0^1$,

$$G_2(\omega_2, \boldsymbol{q}_0^1, \overline{\boldsymbol{q}}_0^1, \boldsymbol{q}_1^0, \boldsymbol{q}_1^2, \overline{\boldsymbol{q}}_1^2)$$

can be written as $N_2(\omega_2, \boldsymbol{q}_0^1, \overline{\boldsymbol{q}}_0^1)$. Since ω_0 is the eigenvalue for bending waves, a solvability condition must be envoked to remove secularity in $\boldsymbol{q}_2^1$. This condition determines ω_2. For the case considered, ω_2 is found to be negative imaginary; the flow is then unstable.

This procedure is very similar to non-linear stability analysis but our solution is an expansion in ϵ, the parameter governing the distortion of the mean flow of a line filament into the ring flow field rather than an expansion in the amplitude of the wave disturbances. We have retained only terms linear in amplitude in this analysis. The problem considered is the stability of a flow which differs from that of the basic flow (the line filament) by an asymptotic solution in ϵ, the ratio of vortex-core size to radius of curvature. To include finite amplitude effects would require an expansion in both ϵ and amplitude.

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In this paper, we consider only the possible instabilities at wavenumbers for which $\omega_0 = 0$ and we will present numerical results for only the lowest two values of k for which $\omega_0 = 0$ is an eigenvalue (these are the bending waves with the second and third radial displacement mode in which the various radial stations of the core move in opposition). For a vortex without axial flow, all wavenumbers for which $\omega_0 = 0$ satisfy the condition $\omega_0^1 = \varpi_0^1$ since in this case the dispersion curves are related by a reflection ($\varpi_0^1 = -\omega_0^1$).

We shall not consider the case $k = 0$, $\omega_0 = 0$, a simple translation of the core. A line filament in a strain would be unstable to this displacement. However, a vortex pair would not be unstable to this mode nor is the ring unstable; growth of this mode requires an increase in the impulse of the flow with time.

For the line vortex, in the presence of strain, the most unstable wave at finite k is the second radial bending mode, the first crossing of the dispersion curve at finite k (Tsai & Widnall 1976). For the vortex ring, good agreement has been obtained with the experimental observations of the unstable wavenumber by applying the simple criterion $\omega_0 = 0$ and using the first finite wave number k for which this occurs (Widnall *et al.* 1974). Recent experiments by Maxworthy (1976) have also indicated an instability at about twice this wavenumber. This could be the wavenumber for which the dispersion relation for the third radial bending mode crosses $\omega_0 = 0$ ($k \sim 4.35$, see figure 4) which occurs at a wavenumber roughly twice that of the second radial mode.

6. The lowest-order solution: waves on a straight filament with constant vorticity

To lowest order, the governing disturbance equations (2.8) and associated boundary conditions (2.11, 2.13) are that for the linear homogeneous eigenvalue problem of waves on a straight filament

for $r < 1$

$$\mathrm{i}\omega_0 u_0 + \frac{\partial u_0}{\partial \theta} - 2v_0 + \frac{\partial \pi_0}{\partial r} = 0, \tag{6.1a}$$

$$\mathrm{i}\omega_0 v_0 + \frac{\partial v_0}{\partial \theta} + 2u_0 + \frac{1}{r}\frac{\partial \pi_0}{\partial \theta} = 0, \tag{6.1b}$$

$$\mathrm{i}\omega_0 w_0 + \frac{\partial w_0}{\partial \theta} + \mathrm{i}k\pi_0 = 0, \tag{6.1c}$$

$$\frac{\partial u_0}{\partial r} + \frac{u_0}{r} + \frac{1}{r}\frac{\partial v_0}{\partial \theta} + \mathrm{i}kw_0 = 0 \tag{6.1d}$$

for $r > 1$

$$\frac{\partial^2 \phi_0}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_0}{\partial \theta^2} - k^2\phi_0 = 0 \tag{6.1e}$$

with boundary condition at $r = 1$

$$\left.\begin{aligned} u_0 - \mathrm{i}\omega_0 f_0 - \frac{\partial f_0}{\partial \theta} &= 0,\\ \frac{\partial \phi_0}{\partial r} - \mathrm{i}\omega_0 f_0 - \frac{\partial f_0}{\partial \theta} &= 0,\\ p_0 + \mathrm{i}\omega_0 \phi_0 + \frac{\partial \phi_0}{\partial \theta} &= 0.\end{aligned}\right\} \tag{6.2}$$

Although the solution to this problem is well known (Kelvin 1880), we carry it through to introduce the notation and the procedure that will illuminate the structure of the higher-order non-homogeneous problems. Although we are concerned primarily with the instability of bending waves, the structure of the problems for all wave types is of interest for the higher-order solutions. We shall, therefore, assume a general solution to (6.1, 6.2) of the form

$$\begin{aligned} \boldsymbol{q}_0(r,\theta) &= \{u_0, v_0, w_0, \pi_0, \phi_0, f_0\} \\ &= \sum_{m=0}^{\infty} \{u_0^m(r), v_0^m(r), w_0^m(r), \pi_0^m(r), \phi_0^m(r), f_0^m\}\,\mathrm{e}^{\mathrm{i}m\theta} \\ &\quad + \sum_{m=1}^{\infty} \{\bar{u}_0^m(r), \bar{v}_0^m(r), \bar{w}_0^m(r), \bar{\pi}_0^m(r), \bar{\phi}_0^m(r), \bar{f}_0^m\}\,\mathrm{e}^{-\mathrm{i}m\theta}, \end{aligned} \tag{6.3}$$

where the subscript 0 denotes the zero order-solution, the superscript m indicates the dependence $m\theta$ and the barred quantities are associated with the $\mathrm{e}^{-\mathrm{i}m\theta}$ dependence.

The equations for the flow inside the core (6.1 *a–d*) can be manipulated into equations for the pressure components of the assumed solutions (6.3), $\pi_0^m(r)$ and $\bar{\pi}_0^m(r)$. The equations governing both the barred and unbarred components can be written

$$\frac{\mathrm{d}^2\pi_0^m}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\pi_0^m}{\mathrm{d}r} + ((\eta_m)^2 - m^2/r^2)\,\pi_0^m = 0, \tag{6.4}$$

where $(\eta_m)^2 = k^2(4-(\omega_0 \pm m)^2)/(\omega_0 \pm m)^2$ with the minus sign taken for $(\bar{\eta}_m)^2$.

The solutions to (6.4) are

$$\left.\begin{aligned} \pi_0^m &= \beta_0^m\, J_m(\eta_m r), \\ \bar{\pi}_0^m &= \bar{\beta}_0^m\, J_m(\bar{\eta}_m r). \end{aligned}\right\} \tag{6.5}$$

The remaining components of the disturbance vector $\boldsymbol{q}_0$ can be found by operations on the pressure. The details are set out in Tsai (1976). We shall introduce the following notation for the solution

$$\left.\begin{aligned} \{u_0^m, v_0^m, w_0^m, \pi_0^m\} &= \{A_0^m, B_0^m, C_0^m, D_0^m\}\,\beta_0^m, \\ \{\bar{u}_0^m, \bar{v}_0^m, \bar{w}_0^m, \bar{\pi}_0^m\} &= \{\bar{A}_0^m, \bar{B}_0^m, \bar{C}_0^m, \bar{D}_0^m\}\,\bar{\beta}_0^m \end{aligned}\right\} \tag{6.6}$$

to separate dependence on wave amplitude. The functions A_0^m, B_0^m, etc., are operations on the Bessel function $J_m(\eta_m r)$, the solution (6.5) for the pressure; in particular, D_0^m and $\bar{D}_0^m$ equal $J_m(\eta_m r)$ and $J_m(\bar{\eta}_m r)$. These solutions can be obtained by standard techniques.

For the disturbance outside the core, the governing equation for the components of the assumed form of ϕ_0^m (6.3) is

$$\frac{\mathrm{d}^2\phi_0^m}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi_0^m}{\mathrm{d}r} - \left(\frac{m^2}{r^2} + k^2\right)\phi_0^m = 0 \tag{6.7}$$

for both ϕ_0^m and $\bar{\phi}_0^m$. The solutions to these equations are

$$\left.\begin{aligned} \phi_0^m &= \alpha_0^m K_m(kr), \\ \bar{\phi}_0^m &= \bar{\alpha}_0^m K_m(kr). \end{aligned}\right\} \tag{6.8}$$

With the assumed form of solution (6.3), the boundary conditions at $r = 1$ become

$$\mathrm{i}(\omega_0 \pm m)\, f_0^m - u_0^m = 0, \tag{6.9a}$$

$$u_0^m - \mathrm{d}\phi_0^m/\mathrm{d}r = 0, \tag{6.9b}$$

$$\pi_0^m + \mathrm{i}(\omega_0 \pm m)\,\phi_0^m = 0. \tag{6.9c}$$

The boundary conditions (6.9*b*, *c*) are used to determine the relations between wave amplitudes α_0^m and β_0^m and between $\bar{\alpha}_0^m$ and $\bar{\beta}_0^m$; f_0^m and $\bar{f}_0^m$ can be determined separately from (6.9*a*).

This procedure gives two linear homogeneous algebraic equations relating α_0^m and β_0^m for each wave type. Using the notation $u_0^m = A_0^m \beta_0^m$ introduced in (6.6), and the solution (6.5) for pressure and (6.8) for velocity potential, we obtain (6.10) directly from (6.9*b*, *c*). Written in matrix notation, the boundary conditions become

$$\begin{bmatrix} -kK_m'(k) & A_0^m \\ \mathrm{i}(\omega_0+m)\,K_m(k) & J_m(\eta_m) \end{bmatrix} \begin{Bmatrix} \alpha_0^m \\ \beta_0^m \end{Bmatrix} = 0, \tag{6.10a}$$

$$\begin{bmatrix} -kK_m'(k) & \bar{A}_0^m \\ \mathrm{i}(\omega_0-m)\,K_m(k) & J_m(\bar{\eta}_m) \end{bmatrix} \begin{Bmatrix} \bar{\alpha}_0^m \\ \bar{\beta}_0^m \end{Bmatrix} = 0, \tag{6.10b}$$

where

$$A_0^m = -\mathrm{i}[-(\omega_0+m)\,\eta_m\,J_0(\eta_m\,r) + (\omega_0-m)\,J_1(\eta_m\,r)/r]/[(\omega_0-m)\,(\omega_0+3m)]_{r=1} \tag{6.10c}$$

and

$$\bar{A}_0^m = -\mathrm{i}[-(\omega_0-m)\,\bar{\eta}_m\,J_0(\bar{\eta}_m\,r) + (\omega_0+m)\,J_1(\bar{\eta}_m\,r)/r]/[(\omega_0+m)\,(\omega_0-3m)]_{r=1}. \tag{6.10d}$$

(The wavenumber k has been taken as positive.)

For a non-trivial solution of (6.10), the determinates must be zero. This requirement gives, of course, the various dispersion relations for waves on a line filament. The eigenvalues ω_0^m and $\bar{\omega}_0^m$ for which the determinates are zero are all real.

The roots of the dispersion relation for a vortex without axial flow for $m = 1$ are shown in figure 4. From (6.4) and (6.10) it can be seen that the dispersion relation for the $\mathrm{e}^{-\mathrm{i}m\theta}$ waves can be obtained by a reflexion of the dispersion relation for the $\mathrm{e}^{\mathrm{i}m\theta}$ waves about $\omega_0^m = 0$. For this homogeneous problem, if ω_0 is not equal to the eigenvalue of the determinant corresponding to a particular wave, then the amplitude of this wave as determined by (6.10) is zero. In determining the higher-order solutions for waves on a curved filament (§§ 7, 8), we shall be solving non-homogeneous equations with the same determinates as (6.10) for the wave amplitudes α_j^m, β_j^m. In that case, the wave amplitudes are easily determined whenever the determinate of (6.10) is *not* zero. When the determinant is zero, a solvability condition must be applied. This solvability condition determines the correction to the frequency of the wave due to curvature of the filament.

The solution of (6.10) also determines the proportionality between α_0^m and β_0^m and between $\bar{\alpha}_0^m$ and $\bar{\beta}_0^m$ for the $\mathrm{e}^{\mathrm{i}m\theta}$ and $\mathrm{e}^{-\mathrm{i}m\theta}$ modes.

$$\alpha_0^m = \frac{A_0^m}{kK_m'(k)}\,\beta_0^m; \quad \bar{\alpha}_0^m = \frac{\bar{A}_0^m}{kK_m'(k)}\,\bar{\beta}_0^m. \tag{6.11}$$

Having given the general solution to the homogeneous equations for waves on a straight filament, we now restrict our consideration to the zero-order solution for the two possible *bending* waves ($\mathrm{e}^{+\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$) since we are interested primarily in the instability of bending waves on the vortex ring. From (6.3), (6.5) and (6.8), the zero-order solution can be written

$$\begin{aligned} \boldsymbol{q}_0 &= \{u_0, v_0, w_0, \pi_0, \phi_0, f_0\} \\ &= \{A_0, B_0, C_0, D_0, E_0, F_0\}\,\beta_0\,\mathrm{e}^{\mathrm{i}\theta} + \{\bar{A}_0, \bar{B}_0, \bar{C}_0, \bar{D}_0, \bar{E}_0, \bar{F}_0\}\,\bar{\beta}_0\,\mathrm{e}^{-\mathrm{i}\theta}, \end{aligned} \tag{6.12}$$

where all symbols in (6.12) have previously been defined except for E_0, $\bar{E}_0$ which from (6.8) and (6.11) equal

$$E_0 = K_1(kr)\,\frac{A_0}{kK_1'(k)}; \quad \bar{E}_0 = K_1(kr)\,\frac{\bar{A}_0}{kK_1'(k)}. \tag{6.13}$$

The superscript 1 identifying the zero-order solution as a bending wave has been dropped wherever possible. The dispersion relations for these modes are

$$\left.\begin{aligned} \begin{vmatrix} -kK_1'(k) & A_0 \\ \mathrm{i}(\omega_0+1)\,K_1(k) & J_1(\eta_1) \end{vmatrix} &= 0 \\ \text{and} \qquad \begin{vmatrix} -kK_1'(k) & \bar{A}_0 \\ \mathrm{i}(\omega_0-1)\,K_1(k) & J_1(\bar{\eta}_1) \end{vmatrix} &= 0 \end{aligned}\right\} \tag{6.14}$$

with A_0, $\bar{A}_0$ defined by (6.10*c*, *d*). The roots of the dispersion relations for these bending waves are shown in figure 4.

There are many possible modes of bending of a line filament. In addition to the bending mode in which all radial stations of the core move in the same direction (the first radial mode), there are an infinite number of higher modes in which the initially concentric stations of the core move in opposition.

At a general ω_0, k point there are three possibilities: if the point does not lie on one of the dispersion curves, there is no solution to the homogeneous problem; if ω_0, k is on one of the dispersion curves, the wave amplitude of that solution is arbitrary and the amplitude of the other wave is zero; if ω_0, k lies at an intersection point of the two families of dispersion relations, both waves can simultaneously exist at arbitrary amplitudes. For a vortex line in the presence of strain, it is a necessary but not sufficient condition for instability that ω_0, k for the isolated filament be located at one of these crossings. The physical significance of these crossing points is that the eigenmodes ($e^{i\theta}$ and $e^{-i\theta}$) of the same wavenumber k and frequency ω_0 can be combined to produce a standing wave displacement of the core boundary of constant angular orientation. If the crossing occurs at $\omega_0 = 0$, this will be a steady displacement wave; if at finite ω_0, the boundary of the core will oscillate as a standing wave of frequency ω_0, but the wave will not change its angular orientation. (However, the unsteady flow inside the core will not have this character unless the two disturbance mode shapes are identical within the core.)

To lowest order, we have found the mode shape and frequencies of bending waves on the vortex ring. Since this problem is identical to the homogeneous problem for waves on an isolated straight filament, these waves are all stable.

We now continue to determine the first-order effects in ϵ upon the frequency and mode shape of these bending waves. We will refer often to the structure of the problem outlined in this section.

7. The first-order solution: waves on a curved filament

From (2.8), the governing disturbance equations for the $O(\epsilon)$ effects of curvature on the bending waves $\boldsymbol{q}_1$ are –

for $r < 1$:

$$\mathrm{i}\omega_0 u_1 + \frac{\partial u_1}{\partial \theta} - 2v_1 + \frac{\partial \pi_1}{\partial r} = -\left(\mathrm{i}\omega_1 + \frac{\partial U_1}{\partial r}\right) u_0 - U_1 \frac{\partial u_0}{\partial r} - \frac{V_1}{r}\frac{\partial u_0}{\partial \theta} - \left(\frac{1}{r}\frac{\partial U_1}{\partial \theta} - \frac{2V_1}{r}\right) v_0, \tag{7.1a}$$

$$\mathrm{i}\omega_0 v_1 + 2u_1 + \frac{\partial v_1}{\partial \theta} + \frac{1}{r}\frac{\partial \pi_1}{\partial \theta} = -\left(\mathrm{i}\omega_1 + \frac{1}{r}\frac{\partial V_1}{\partial \theta} + \frac{U_1}{r}\right) v_0 - \left(\frac{\partial V_1}{\partial r} + \frac{V_1}{r}\right) u_0 - U_1 \frac{\partial v_0}{\partial r} - \frac{V_1}{r}\frac{\partial v_0}{\partial \theta}, \tag{7.1b}$$

$$\mathrm{i}\omega_0 w_1 + \frac{\partial w_1}{\partial \theta} + \mathrm{i}k\pi_1 = -(\mathrm{i}\omega_1 + r\cos\theta)\, w_0 - \frac{V_1}{r}\frac{\partial w_0}{\partial \theta} - U_1 \frac{\partial w_0}{\partial r} + \mathrm{i}kr\sin\theta\,\pi_0, \tag{7.1c}$$

$$\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{1}{r}\frac{\partial v_1}{\partial \theta} + \mathrm{i}kw_1 = -\sin\theta\, u_0 - \cos\theta\, v_0 + \mathrm{i}kr\sin\theta\, w_0; \tag{7.1d}$$

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for $r > 1$:

$$\frac{\partial^2\phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi_1}{\partial\theta^2}-k^2\phi_1 = -\sin\theta\frac{\partial\phi_0}{\partial r}-\frac{\cos\theta}{r}\frac{\partial\phi_0}{\partial\theta}-2k^2r\sin\theta\phi_0, \tag{7.1e}$$

with the boundary conditions at $r = 1$,

$$\left.\begin{aligned} \mathrm{i}\omega_0 f_1+\frac{\partial f_1}{\partial\theta}-u_1 &= -\mathrm{i}\omega_1 f_0-V_1\frac{\partial f_0}{\partial\theta}+\frac{\partial U_1}{\partial r}f_0,\\ u_1-\frac{\partial\phi_1}{\partial r} &= 0,\\ \pi_1+\mathrm{i}\omega_0\phi_1+\frac{\partial\phi_1}{\partial\theta} &= -\mathrm{i}\omega_1\phi_0-\frac{\partial\Phi_1}{\partial\theta}\frac{\partial\phi_0}{\partial\theta}. \end{aligned}\right\} \tag{7.2}$$

The important features of these equations are as follows:

(1) They are non-homogeneous with forcing terms which are the product of the lower-order solution $\boldsymbol{q}_0$ from (6.12) and either the $O(\epsilon)$ mean flow (4.22) or ω_1.

(2) The governing equation for $\boldsymbol{q}_1$ and the associated boundary conditions are identical to those for the zero-order problem except for these non-homogeneous terms.

(3) An important difference between (6.1) and (7.1) is that ω_0 is no longer an unknown eigenvalue to be determined in the solution; ω_0 is a root of the dispersion relation for waves on a straight filament.

(4) The unknown frequency ω_1 appears in the non-homogeneous forcing terms.

Because of the θ-dependence of both the mean flow and the lowest-order solution $\boldsymbol{q}_0$, the governing equations and boundary conditions contain non-homogeneous terms with θ-dependence 1, $\mathrm{e}^{\pm\mathrm{i}\theta}$, $\mathrm{e}^{\pm 2\mathrm{i}\theta}$. We therefore assume a solution of the following form for the extended disturbance vector

$$\begin{aligned} \boldsymbol{q}_1(r,\theta) = {} & \{u_1^0(r), v_1^0(r), w_1^0(r), \pi_1^0(r), \phi_1^0(r), f_1^0\}\\ & +\{u_1^1(r), v_1^1(r), w_1^1(r), \pi_1^1(r), \phi_1^1(r), f_1^1\}\,\mathrm{e}^{\mathrm{i}\theta}\\ & +\{\bar{u}_1^1(r), \bar{v}_1^1(r), \bar{w}_1^1(r), \bar{\pi}_1^1(r), \bar{\phi}_1^1(r), \bar{f}_1^1\}\,\mathrm{e}^{-\mathrm{i}\theta}\\ & +\{u_1^2(r), v_1^2(r), w_1^2(r), \pi_1^2(r), \phi_1^2(r), f_1^2\}\,\mathrm{e}^{2\mathrm{i}\theta}\\ & +\{\bar{u}_1^2(r), \bar{v}_1^2(r), \bar{w}_1^2(r), \bar{\pi}_1^2(r), \bar{\phi}_1^2(r), \bar{f}_1^2\}\,\mathrm{e}^{-2\mathrm{i}\theta}. \end{aligned} \tag{7.3}$$

Note that the subscript 1 refers to the first-order solution, the superscript m refers to the dependence on $m\theta$ and the barred quantities denote $\mathrm{e}^{-\mathrm{i}m\theta}$ dependence.

When this assumed form for $\boldsymbol{q}_1(r, \theta)$ is used in the governing equations (7.1) and boundary conditions (7.2), the following set of equations are obtained for $r < 1$:

$$\left.\begin{aligned} \mathrm{i}(\omega_0\pm m)\,u_1^m-2v_1^m+\frac{\mathrm{d}\pi_1^m}{\mathrm{d}r} &= F_1^m,\\ 2u_1^m+\mathrm{i}(\omega_0\pm m)\,v_1^m\pm\frac{\mathrm{i}m}{r}\pi_1^m &= F_2^m,\\ \mathrm{i}(\omega_0\pm m)\,w_1^m+\mathrm{i}k\pi_1^m &= F_3^m,\\ \frac{\mathrm{d}u_1^m}{\mathrm{d}r}+\frac{u_1^m}{r}\pm\frac{\mathrm{i}m}{r}v_1^m+\mathrm{i}kw_1^m &= F_4^m; \end{aligned}\right\} \tag{7.4}$$

for $r > 1$:

$$\frac{\mathrm{d}^2\phi_1^m}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\phi_1^m}{\mathrm{d}r}-\left(\frac{m^2}{r^2}+k^2\right)\phi_1^m = F_5^m,$$

with boundary conditions at $r = 1$

$$\left.\begin{aligned} \mathrm{i}(\omega_0 \pm m) f_1^m - u_1^m &= B_1^m, \\ u_1^m - \mathrm{d}\phi_1^m/\mathrm{d}r &= 0, \\ \pi_1^m + \mathrm{i}(\omega_0 \pm m)\,\phi_1^m &= B_2^m, \end{aligned}\right\} \tag{7.5}$$

where m takes values 0, 1, and 2 for the various modes in (7.3), and the minus sign in $\pm$ is to be used in the equation for any of the barred variables ($\bar{u}$, $\bar{v}$, etc.). The forcing functions $F_j^m(r)$ and the non-homogeneous terms in the boundary condition can be obtained by the indicated operations on the solution for $\boldsymbol{q}_0$ in terms of Bessel functions and the mean flow solution, $\boldsymbol{Q}_1$.

From § 3, we know that the appropriate boundary condition at $r \to \infty$ is that the homogeneous solution goes to zero as $r \to \infty$.

The forcing terms in F_j^m and the non-homogeneous terms in the boundary conditions, B_j^m, are linear in wave amplitude β_0 or $\bar{\beta}_0$. In addition, only those non-homogeneous terms associated with the $\mathrm{e}^{\pm \mathrm{i}\theta}$ waves contain ω_1; ω_1 multiplies *every* non-homogeneous term in both the governing equations and associated boundary conditions for these modes. Thus, $F_j^1(r)$ can be rewritten as $\omega_1 F_j^1(r)$ with a corresponding change in the meaning of $F_j^1(r)$. $B_j^1(7.5)$ can likewise be redesignated as $\omega_1 B_j^1$. For $r < 1$, the governing equations (7.4*a–d*) can be manipulated into non-homogeneous equations for the pressure of the form

$$\left.\begin{aligned} \frac{\mathrm{d}^2\pi_1^0}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\pi_1^0}{\mathrm{d}r} + (\eta_0)^2\pi_1^0 &= T_1^0(r)\,\beta_0 - \bar{T}_1^0(r)\,\bar{\beta}_0, \\ \frac{\mathrm{d}^2\pi_1^1}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\pi_1^1}{\mathrm{d}r} + \left[\frac{-1}{r^2} + (\eta_1)^2\right]\pi_1^1 &= -\mathrm{i}\omega_1\,T_1^1(r)\,\beta_0, \\ \frac{\mathrm{d}^2\bar{\pi}_1^1}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\bar{\pi}_1^1}{\mathrm{d}r} + \left[\frac{-1}{r^2} + (\bar{\eta}_1)^2\right]\bar{\pi}_1^1 &= -\mathrm{i}\omega_1\,\bar{T}_1^1(r)\,\bar{\beta}_0, \\ \frac{\mathrm{d}^2\pi_1^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\pi_1^2}{\mathrm{d}r} + \left[\frac{-4}{r^2} + (\eta_2)^2\right]\pi_1^2 &= T_1^2(r)\,\beta_0, \\ \frac{\mathrm{d}^2\bar{\pi}_1^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\bar{\pi}_1^2}{\mathrm{d}r} + \left[\frac{-4}{r^2} + (\bar{\eta}_2)^2\right]\bar{\pi}_1^2 &= \bar{T}_1^2(r)\,\bar{\beta}_0, \end{aligned}\right\} \tag{7.6}$$

where as before $(\eta_m)^2$ or $(\bar{\eta}_m)^2 = k^2(4-(\omega_0 \pm m)^2)/(\omega_0 \pm m)^2$ with the minus sign taken for $\bar{\eta}_m$.

The notation $T_1^m(r)\,\beta_0$ has been introduced to show explicitly the dependence of the non-homogeneous terms on wave amplitudes β_0 and $\bar{\beta}_0$.

For $r > 1$, the equations governing ϕ_1^m become

$$\left.\begin{aligned} \frac{\mathrm{d}^2\phi_1^0}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi_1^0}{\mathrm{d}r} - k^2\phi_1^0 &= R_1^0(r)\,(\beta_0 - \bar{\beta}_0), \\ \frac{\mathrm{d}^2\phi_1^1}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi_1^1}{\mathrm{d}r} - \left(\frac{1}{r^2} + k^2\right)\phi_1^1 &= 0, \quad \text{for both } \phi_1^1 \text{ and } \bar{\phi}_1^1, \\ \frac{\mathrm{d}^2\phi_1^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi_1^2}{\mathrm{d}r} - \left(\frac{4}{r^2} + k^2\right)\phi_1^2 &= \begin{cases} R_1^2\,\beta_0 & \text{for } \phi_1^2 \\ -R_1^2\,\bar{\beta}_0 & \text{for } \bar{\phi}_1^2 \end{cases}. \end{aligned}\right\} \tag{7.7}$$

Again, the forcing terms are linearly proportional to wave amplitudes β_0 and $\bar{\beta}_0$. The forcing terms T_1^0, R_1^0, etc., contain Bessel functions and polynomials in r; the forcing terms are, in general, complex with various symmetries determined by the form of the lowest-order wave solutions and the mean flow field.

Although tedious, it is quite straightforward to obtain the solution to these equations. We shall, therefore, omit the uninteresting analytical details that are required to solve this problem and instead concentrate on the structure of the problem, the form of the solution and the major conclusions that follow. We shall also present the numerical results of the amplification rate for the particular case considered which is obtained by actually carrying out the solution in detail. (The details of the solution are available in Tsai (1976).)

We now consider the solution only for the case $\omega_0 = 0$. As previously mentioned, this is the very special case for which waves on a line filament will not rotate. By our previous work (Tsai & Widnall 1976) we know that this condition satisfies the necessary condition for instability of a line filament in the presence of strain. Also, since $\omega_0 = 0$ is a root of the dispersion relation and its reflexion, at this condition both wave amplitudes β_0 and $\bar{\beta}_0$ are non-zero and arbitrary. This particular case gives well defined symmetries to the solution. For $m = 0$ and $m = 2$, ω_0 is not an eigenvalue of the homogeneous form of the equations (7.6).

The solution to (7.6) consists of a homogeneous solution identical in form to (6.6) plus particular solutions linearly proportional to β_0 and $\bar{\beta}_0$. In general, this solution is of the form

$$\begin{aligned}\{u_1^m, v_1^m, w_1^m, \pi_1^m, f_1^m\} &= \{A_0^m, B_0^m, C_0^m, D_0^m, F_0^m\}\beta_1^m \\ &\quad + \{U^m, V_1^m, W_1^m, \Pi_1^m, F_1^m\}\beta_0 \\ &\quad + \{\bar{U}_1^m, \bar{V}_1^m, \bar{W}_1^m, \bar{\Pi}_1^m, \bar{F}_1^m\}\bar{\beta}_0,\end{aligned} \tag{7.8}$$

where $U_1^m \ldots$, $\bar{U}_1^m \ldots$ denote the particular solutions from the non-homogeneous terms proportional to β_0 and $\bar{\beta}_0$, and A_0^m, etc., are the solutions to the homogeneous equation, which therefore have the same meaning as in equation (6.6).

We can write a more specific form for the particular solutions in (7.8) by taking advantage of the symmetries of the particular solution for $\omega_0 = 0$ and the fact that each equation in (7.6) does not contain both β_0 and $\bar{\beta}_0$.

Specifically, the particular solutions are

$$\left.\begin{aligned}\{u_1^0, v_1^0, w_1^0, \pi_1^0\}_p &= \{U_1^0, V_1^0, W_1^0, \Pi_1^0\}\beta_0 + \{U_1^0, -V_1^0, W_1^0, -\Pi_1^0\}\bar{\beta}_0, \\ \{u_1^1, v_1^1, w_1^1, \pi_1^1\}_p &= \omega_1\{U_1^1, V_1^1, W_1^1, \Pi_1^1\}\beta_0, \\ \{\bar{u}_1^1, \bar{v}_1^1, \bar{w}_1^1, \bar{\pi}_1^1\}_p &= \omega_1\{U_1^1, -V_1^1, W_1^1, -\Pi_1^1\}\bar{\beta}_0, \\ \{u_1^2, v_1^2, w_1^2, \pi_1^2\}_p &= \{U_1^2, V_1^2, W_1^2, \Pi_1^2\}\beta_0, \\ \{\bar{u}_1^2, \bar{v}_1^2, \bar{w}_1^2, \bar{\pi}_1^2\}_p &= \{U_1^2, -V_1^2, W_1^2, -\Pi_1^2\}\bar{\beta}_0,\end{aligned}\right\} \tag{7.9}$$

where the solutions U_1^0, etc., are functions of r obtained by finding the particular solution to the equations (7.6) for the pressure and from this determining the other components of the disturbance field. Again, we shall not give the explicit forms for these solutions since they can be obtained by standard techniques.

What is significant are the symmetries in the solutions, the linear dependence on β_0 and $\bar{\beta}_0$ and the fact that the particular solution for $\{u_1^1 \ldots\}$ (or $\{\bar{u}_1^1 \ldots\}$) is linear in ω_1 and depends only on β_0 (or $\bar{\beta}_0$).

Similarly, the solution to (7.7) is of the form

$$\left.\begin{aligned}\phi_1^0 &= \alpha_1^0 K_0(kr) + G_0(\beta_0 - \bar{\beta}_0), \\ \phi_1^1 &= \alpha_1^1 K_1(kr); \quad \bar{\phi}_1^1 = \bar{\alpha}_1^1 K_1(kr), \\ \phi_1^2 &= \alpha_1^2 K_2(kr) + G_2\beta_0; \quad \bar{\phi}_1^2 = \bar{\alpha}_1^2 K_2(kr) - G_2\bar{\beta}_0,\end{aligned}\right\} \tag{7.10}$$

where $\phi_1^m \to 0$ at $r \to \infty$.

When the solutions (7.8) [with (7.9)] and (7.10) are substituted into the boundary conditions (7.5*b*, *c*), we obtain two non-homogeneous linear algebraic equations relating α_1^m and β_1^m and relating $\bar{\alpha}_1^m$ and $\bar{\beta}_1^m$ for each m. These equations are analogous to equation (6.10) for the related homogeneous eigenvalue problems for waves on the straight filament.

Even though we have chosen $\omega_0 = 0$, we will write these equations in the form of (6.10) to illuminate the structure of the problem.

For $m = 2$ we have

$$\begin{vmatrix} -kK_2'(k) & A_0^2 \\ \mathrm{i}(\omega_0+2)\,K_2(k) & J_2(\eta_2) \end{vmatrix} \begin{bmatrix} \alpha_1^2 \\ \beta_1^2 \end{bmatrix} = \begin{bmatrix} Q_1^2 \\ R_1^2 \end{bmatrix} \beta_0, \tag{7.11a}$$

$$\begin{vmatrix} -kK_2'(k) & \bar{A}_0^2 \\ \mathrm{i}(\omega_0-2)\,K_2(k) & J_2(\bar{\eta}_2) \end{vmatrix} \begin{Bmatrix} \bar{\alpha}_1^2 \\ \bar{\beta}_1^2 \end{Bmatrix} = \begin{Bmatrix} +Q_1^2 \\ -R_1^2 \end{Bmatrix} \bar{\beta}_0, \tag{7.11b}$$

where the non-homogeneous terms Q_1^m and R_1^m come both from the particular solution evaluated at $r = 1$ and from the non-homogeneous terms in the boundary conditions. Again, these terms involve various Bessel functions and polynomials in r and are straightforward but tedious to obtain. For $m = 2$, the determinants of (7.11) are not zero when $\omega_0 = 0$, since $\omega_0 = 0$ is not an eigenvalue for this value of k. Therefore, α_1^2, β_1^2 and $\bar{\alpha}_1^2$, $\bar{\beta}_1^2$ are simply found and are linearly proportional to β_0 and $\bar{\beta}_0$ (which at this point are still arbitrary).

For $m = 0$, $\omega_0 = 0$ is not an eigenvalue but there is a degeneracy in the equations, which must be treated with care. For $\omega_0 = 0$, the dependence on the $m = 0$ (bulge) wave amplitude on the lowest-order bending wave amplitudes β_0 and $\bar{\beta}_0$ is of the form

$$\beta_1^0 = \mathscr{B}_1^0(\beta_0 - \bar{\beta}_0); \quad \alpha_1^0 = \mathscr{A}_1^0(\beta_0 - \bar{\beta}_0),$$

where $\mathscr{B}_1^0$ and $\mathscr{A}_1^0$ are functions of k only, involving Bessel functions.

The dependence of the $m = 2$ (shape-change) wave amplitudes are of the form –

for dependence $e^{2\mathrm{i}\theta}$: $\quad \beta_1^2 = \mathscr{B}_1^2\,\beta_0; \quad \alpha_1^2 = \mathscr{A}_1^2\,\beta_0;$

and for dependence $e^{-2\mathrm{i}\theta}$: $\quad \bar{\beta}_1^2 = +\mathscr{B}_1^2\bar{\beta}_0; \quad \bar{\alpha}_1^2 = +\mathscr{A}_1^2\,\bar{\beta}_0,$

where $\mathscr{B}_1^2$ and $\mathscr{A}_1^2$ are also functions of k only, involving Bessel functions.

These solutions show that, although to lowest order we are considering bending waves on the vortex ring, the actual form of the waves involves both bulge modes and shape changes at $O(\epsilon)$ due to the effects of curvature. The amplitudes of these induced waves are proportional to the amplitudes of the bending waves β_0 and $\bar{\beta}_0$.

For $m = 1$, we have a very different problem. The equation governing the amplitudes of the $O(\epsilon)$ bending waves is

$$\left.\begin{aligned} &\begin{vmatrix} -kK_1'(k) & A_0^1 \\ \mathrm{i}(\omega_0+1)\,K_1(k) & J_1(\eta_1) \end{vmatrix} \begin{Bmatrix} \alpha_1^1 \\ \beta_1^1 \end{Bmatrix} = \beta_0\,\omega_1 \begin{Bmatrix} Q_1^1 \\ R_1^1 \end{Bmatrix}, \\ &\begin{vmatrix} -kK_1'(k) & \bar{A}_0^1 \\ \mathrm{i}(\omega_0-1)\,K_1(k) & J_1(\bar{\eta}_1) \end{vmatrix} \begin{Bmatrix} \bar{\alpha}_1^1 \\ \bar{\beta}_1^1 \end{Bmatrix} = \bar{\beta}_0\,\omega_1 \begin{Bmatrix} \bar{Q}_1^1 \\ \bar{R}_1^1 \end{Bmatrix}. \end{aligned}\right\} \tag{7.12}$$

The determinants of (7.12) are exactly those for the homogeneous problem of bending waves on a straight filament (6.14), and are both equal to zero since ω_0 was chosen as the eigenvalue for both the $e^{\mathrm{i}\theta}$ and $e^{-\mathrm{i}\theta}$ modes. The unknown correction to the frequency ω_1 multiplies the non-homogeneous terms. In general, the solvability of an equation with a zero determinant requires that the forcing terms be orthogonal to the solution of the adjoint homogeneous problem. The general solvability condition is applied in Section 8 where the solution to $O(\epsilon^2)$ is considered. For (7.12) to have a solution, this condition requires that $\omega_1 = 0$. For $\omega_1 = 0$, β_1^1 and $\bar{\beta}_1^1$ are arbitrary and β_0 and $\bar{\beta}_0$ remain arbitrary.

Thus, to $O(\epsilon)$, we have determined the following:

(1) The correction to the frequency ω_1 is zero; there is no destabilizing effect to this order.

(2) Bulge waves and shape-change waves proportional to the bending wave amplitudes are induced by the effects of curvature.

8. The solution to second order: the instability of waves on the ring

The governing disturbance equations for the $O(\epsilon^2)$ effects of curvature on the bending waves $\boldsymbol{q}$ are obtained from (2.8). The $O(\epsilon^2)$ non-homogeneous terms in (2.8*a*, …, *e*) will be denoted by F_i, $i = 1, 5$ –

for $r < 1$:

$$\mathrm{i}\omega_0 u_2 + \frac{\partial u_2}{\partial \theta} - 2v_2 + \frac{\partial \pi_2}{\partial r} = F_1, \tag{8.1a}$$

$$\mathrm{i}\omega_0 v_2 + \frac{\partial v_2}{\partial \theta} + 2u_2 + \frac{1}{r}\frac{\partial \pi_2}{\partial \theta} = F_2, \tag{8.1b}$$

$$\mathrm{i}\omega_0 w_2 + \frac{\partial w_2}{\partial \theta} + \mathrm{i}k\pi_2 = F_3, \tag{8.1c}$$

$$\frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{1}{r}\frac{\partial v_2}{\partial \theta} + \mathrm{i}kw_2 = F_4; \tag{8.1d}$$

for $r > 1$:

$$\frac{\partial^2 \phi_2}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_2}{\partial r} - \left(\frac{1}{r^2}\frac{\partial^2 \phi_2}{\partial \theta^2} + k^2\phi_2\right) = F_5, \tag{8.1e}$$

with boundary conditions at $r = 1$:

$$\mathrm{i}\omega_0 f_2 + \frac{\partial f_2}{\partial \theta} - u_2 = -\mathrm{i}\omega_1 f_1 - V_1\frac{\partial f_1}{\partial \theta} + \frac{\partial U_1}{\partial r}f_1 - \mathrm{i}\omega_2 f_0 - V_2\frac{\partial f_0}{\partial \theta} + \frac{\partial U_2}{\partial r}f_0 + \alpha_2\frac{\partial u_0}{\partial r} - \frac{\mathrm{d}\alpha_2}{\mathrm{d}\theta}v_0, \tag{8.2a}$$

$$u_2 - \frac{\partial \phi_2}{\partial r} = \alpha_2\left(\frac{\partial^2 \phi_0}{\partial r^2} - \frac{\partial u_0}{\partial r}\right) + \frac{\mathrm{d}\alpha_2}{\mathrm{d}\theta}\left(-\frac{\partial \phi_0}{\partial \theta} + v_0\right), \tag{8.2b}$$

$$\pi_2 + \mathrm{i}\omega_0\phi_2 + \frac{\partial \phi_2}{\partial \theta} = -\mathrm{i}\omega_1\phi_1 - \frac{\sin\theta}{4}\frac{\partial \phi_1}{\partial \theta} - \mathrm{i}\omega_2\phi_0 - \frac{\partial \Phi_2}{\partial r}\frac{\partial \phi_0}{\partial r} - \frac{\partial \Phi_2}{\partial \theta}\frac{\partial \phi_0}{\partial \theta}$$
$$+\alpha_2\left(-\frac{\partial \pi_0}{\partial r} - \mathrm{i}\omega_0\frac{\partial \phi_0}{\partial r} + 2\frac{\partial \phi_0}{\partial \theta} - \frac{\partial^2 \phi_0}{\partial \theta\,\partial r}\right). \tag{8.2c}$$

The problem to this order is considerably more complex, but there are several important features worth noting:

(1) Again at this order, the operators on the second-order solution $\boldsymbol{q}_2$ in both the governing equation (8.1) and the related boundary conditions (8.2) are identical to the homogeneous operators for the free bending waves $\boldsymbol{q}_0$. Also, ω_0 has already been identified as an eigenvalue for the homogeneous problem.

(2) The non-homogeneous terms are of three types: products of the bending waves $\boldsymbol{q}_0(\mathrm{e}^{\pm\mathrm{i}\theta})$ with the mean flow at $O(\epsilon^2)$ which is a straining field $(\mathrm{e}^{\pm 2\mathrm{i}\theta})$; products of the modified waves $\boldsymbol{q}_1(1, \mathrm{e}^{\pm\mathrm{i}\theta}, \mathrm{e}^{\pm 2\mathrm{i}\theta})$ with the $O(\epsilon)$ mean flow of the ring $(\mathrm{e}^{\pm\mathrm{i}\theta})$; and the product of the unknown frequency ω_2 with $\boldsymbol{q}_0(\mathrm{e}^{\pm\mathrm{i}\theta})$.

(3) The forcing terms are all linearly proportional to the bending-wave amplitudes β_0 and $\bar{\beta}_0$ except for $\boldsymbol{q}_1$ which contains bending waves of arbitrary amplitude β_1^1 (and $\bar{\beta}_1^1$).

(4) It is possible to separate the forcing terms into those due to curvature and those due to strain in the following way: those terms which would be present in the related problem of the stability of a line filament in the strain field of the *ring* $\boldsymbol{Q}_2$ are assigned to the effects of strain (Tsai & Widnall, 1976); all other terms are assigned to the effects of curvature. These terms are (*a*) the products of $\boldsymbol{q}_1$ with $\boldsymbol{Q}_1$ and (*b*) all non-homogeneous terms in the continuity equation (8.1 *d*, *e*) which is a homogeneous equation in the problem of a line filament in strain. This separation is not physically realizable in that the strain field of the ring is not governed by two-dimensional equations; it does, however, display certain symmetries and may help in the understanding of the instability mechanism.

At this point, we are interested only in determining ω_2, the $O(\epsilon^2)$ correction to the frequency. If (as we shall see) ω_2 has a negative imaginary part, the flow is unstable. Furthermore, we have specialized the analysis to consider the stability only of those bending waves which will be stationary on the line filament ($\omega_0 = 0$). ω_2 appears only in those non-homogeneous terms that are proportional to $\beta_0\,\mathrm{e}^{\mathrm{i}\theta}$ and $\bar{\beta}_0\,\mathrm{e}^{-\mathrm{i}\theta}$. Thus, to determine ω_2, we need only solve the non-homogeneous problem for waves of the form

$$\boldsymbol{q}_2 = \boldsymbol{q}_2^1\,\mathrm{e}^{\mathrm{i}\theta}+\bar{\boldsymbol{q}}_2^1\,\mathrm{e}^{-\mathrm{i}\theta}. \tag{8.3}$$

The lower-order wave solutions (to $O(\epsilon)$) have θ-dependence $\boldsymbol{q}_0 \sim \mathrm{e}^{\pm\mathrm{i}\theta}$ and $\boldsymbol{q}_1 \sim 1, \mathrm{e}^{\pm\mathrm{i}\theta}, \mathrm{e}^{\pm2\mathrm{i}\theta}$; the mean flow to $O(\epsilon^2)$ has θ-dependence $\boldsymbol{Q} \sim 1, \mathrm{e}^{\pm\mathrm{i}\theta}, \mathrm{e}^{\pm\mathrm{i}2\theta}$. Thus, there are many combinations of products of wave solutions with the mean flow that will contribute to the non-homogeneous terms with dependence $\mathrm{e}^{\pm\mathrm{i}\theta}$. These terms can be separated and assigned to strain and curvature as previously discussed.

As before, the governing equations for $r < 1$ can be manipulated into equations for the pressure of the form

$$\left.\begin{aligned}\frac{\mathrm{d}^2\pi_2^1}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\pi_2^1}{\mathrm{d}r}-\frac{\pi_2^1}{r^2}+(\eta_1)^2\pi_2^1 &= \beta_0\,[-\omega_2P_2^{\mathrm{s}}+P_2^{\mathrm{c}}]+\bar{\beta}_0\,[H_2^{\mathrm{c}}+H_2^{\mathrm{s}}],\\ \frac{\mathrm{d}^2\bar{\pi}_2^1}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\bar{\pi}_2^1}{\mathrm{d}r}-\frac{\bar{\pi}_2^1}{r^2}+(\bar{\eta}_1)^2\bar{\pi}_2^1 &= \bar{\beta}_0\,[\omega_2P_2^{\mathrm{s}}+P_2^{\mathrm{c}}]+\beta_0\,[H_2^{\mathrm{c}}+H_0^{\mathrm{s}}],\end{aligned}\right\} \tag{8.4}$$

where the notation used for the non-homogeneous terms indicates (i) the dependence on wave amplitude β_0 and β_0, (ii) the dependence on curvature (superscript c) or strain (superscript s) and (iii) the dependence on the unknown correction to the frequency ω_2. The functions P_2^{s}, P_2^{c}, H_2^{s} and H_2^{c} are complicated functions of k and r involving Bessel functions.

The solutions for π_2^1 and the velocity u_2^1 (which appear in the boundary conditions) are of the form

$$\left.\begin{aligned}\pi_2^1 &= J_1(\eta_1 r)\,\beta_2^1+\beta_0\,[-\omega_2\,\pi_2^{\mathrm{s}}+\pi_2^{\mathrm{c}}]+\bar{\beta}_0\,[\mathscr{L}_2^{\mathrm{c}}+\mathscr{L}_2^{\mathrm{s}}],\\ \bar{\pi}_2^1 &= J_1(\bar{\eta}_1 r)\,\bar{\beta}_2^1+\bar{\beta}_0\,[\omega_2\,\pi_2^{\mathrm{s}}+\pi_2^{\mathrm{c}}]+\beta_0\,[\mathscr{L}_2^{\mathrm{c}}+\mathscr{L}_2^{\mathrm{s}}]\end{aligned}\right\} \tag{8.5}$$

and

$$\left.\begin{aligned}u_2^1 &= A_0\,\beta_2^1+\beta_0\,[\omega_2\,\Omega_2^{\mathrm{s}}+\Omega_2^{\mathrm{c}}]+\bar{\beta}_0\,[\Gamma_2^{\mathrm{c}}+\Gamma_2^{\mathrm{s}}],\\ \bar{u}_2^1 &= \bar{A}_0\,\bar{\beta}_2^1+\bar{\beta}_0\,[\omega_2\,\Omega_2^{\mathrm{s}}-\Omega_2^{\mathrm{c}}]-\beta_0\,[\Gamma_2^{\mathrm{c}}+\Gamma_2^{\mathrm{s}}].\end{aligned}\right\} \tag{8.6}$$

The solution for the velocity potential outside the vortex core is directly obtained from the solution to (8.1 *e*) for the $\mathrm{e}^{\pm\mathrm{i}\theta}$ modes.

$$\left.\begin{aligned}\phi_2^1 &= K_1(kr)\,\alpha_2^1+\beta_0\,G_2^{\mathrm{c}}+\bar{\beta}_0\,H_2^{\mathrm{c}},\\ \bar{\phi}_2^1 &= K_1(kr)\,\bar{\alpha}_2^1-\bar{\beta}_0\,G_2^{\mathrm{c}}-\beta_0\,H_2^{\mathrm{c}}.\end{aligned}\right\} \tag{8.7}$$

The particular solution for the velocity potential is assigned to the effects of curvature since for a line filament in strain, the potential equation is homogeneous. As previously discussed (§3), the velocity potential near the core to $O(\epsilon^2)$ can be obtained by a direct expansion of the external potential solution for waves on a ring expressed in toroidal functions. This has been done and the result is identical to $O(\epsilon^2)$ to the solution of the near field equation (8.1 e). Thus, (8.7) represents the complete potential solution near the core to $O(\epsilon^2)$. Relations between unknown amplitudes of the homogeneous solutions in β_2^1 and α_2^1 (and between $\bar{\beta}_2^1$ and $\bar{\alpha}_2^1$) are found by satisfying the boundary conditions (8.2 b, c) which for the $e^{\pm i\theta}$ modes become

$$\left.\begin{aligned}
u_2^1-\frac{d\phi_2^1}{dr} &= \frac{\alpha_2}{2}\left(\frac{d^2\bar{\phi}_0^1}{dr^2}-\frac{d\bar{u}_0^1}{dr}\right)+i\alpha_2(-i\bar{\phi}_0^1+\bar{v}_0^1),\\
\pi_2^1+i(\omega_0+1)\,\phi_2^1 &= -i\omega_2\,\phi_0^1+\tfrac{1}{4}\phi_1^2+\frac{i}{2}\frac{d\Phi_2}{dr}\frac{d\bar{\phi}_0^1}{dr}+i\Phi_2\,\bar{\phi}_0^1+\frac{\alpha_2}{2}\left[-\frac{d\bar{\pi}_0^1}{dr}-i(\omega_0-1)\frac{d\bar{\phi}_0^1}{dr}-2i\bar{\phi}_0^1\right]\\
\text{and}\quad \bar{u}_2^1-\frac{d\bar{\phi}_2^1}{dr} &= \frac{\alpha_2}{2}\left(\frac{d^2\phi_0^1}{dr^2}-\frac{du_0^1}{dr}\right)-i\alpha_2(-i\phi_0^1+v_0^1),\\
\bar{\pi}_2^1+i(\omega_0-1)\,\bar{\phi}_2^1 &= -i\omega_2\,\bar{\phi}_0^1+\tfrac{1}{4}\bar{\phi}_1^2-\frac{i}{2}\frac{d\Phi_2}{dr}\frac{d\phi_0^1}{dr}-i\Phi_2\,\phi_0^1+\frac{\alpha_2}{2}\left[-\frac{d\pi_0^1}{dr}-i(\omega_0+1)\frac{d\phi_0^1}{dr}+2i\phi_0^1\right],
\end{aligned}\right\} \tag{8.8}$$

where α_2 is the coefficient in $\alpha(\theta) = \epsilon^2\alpha_2\cos 2\theta$ (see (4.23)).

When the solutions (8.5), (8.6) and (8.7) are put into the boundary conditions (8.8), two linear algebraic equations are obtained in the form

$$\begin{bmatrix} -kK_1'(k) & A_0 \\ i(\omega_0+1)\,K_1(k) & J_1(\eta_1) \end{bmatrix}\begin{Bmatrix}\alpha_2^1\\ \beta_2^1\end{Bmatrix} = \begin{bmatrix} -a^s\omega_2+b^c & c \\ d^s\omega_2+e^c & f \end{bmatrix}\begin{Bmatrix}\beta_0\\ \bar{\beta}_0\end{Bmatrix} \tag{8.9a}$$

and

$$\begin{bmatrix} -kK_1'(k) & \bar{A}_0 \\ i(\omega_0-1)\,K_1(k) & J_1(\bar{\eta}_1) \end{bmatrix}\begin{Bmatrix}\bar{\alpha}_2^1\\ \bar{\beta}_2^1\end{Bmatrix} = \begin{bmatrix} a^s\omega_2+b^c & c \\ d^s\omega_2-e^c & -f \end{bmatrix}\begin{Bmatrix}\beta_0\\ \bar{\beta}_0\end{Bmatrix}, \tag{8.9b}$$

where the superscript s or c denotes the effects of strain or curvature. The determinants of the coefficients α_2^1, β_2^1 and $\bar{\alpha}_2^1$, $\bar{\beta}_2^1$ are identically the dispersion relations for waves on the line filament (6.14) and are therefore both identically zero for the case we have considered ($\omega_0 = 0$). At this point in the analysis, the wave amplitudes β_0 and $\bar{\beta}_0$ which appear in the forcing functions are still arbitrary. The correction to the frequency ω_2 is still unknown.

Since the determinants of (8.9 a, b) are zero, a solvability condition must be enforced to obtain a solution. One possibility would be to require β_0 and $\bar{\beta}_0$ to be zero, but the more interesting possibility is to apply the general solvability condition that the forcing terms are orthogonal to the eigenvectors of the adjoint matrix. This condition will determine ω_2 and the relation between β_0 and $\bar{\beta}_0$. (These steps in the analysis are analogous to those leading to (3.18) of Tsai & Widnall 1976.)

This condition can be written in the following form:

$$\begin{bmatrix} \omega_2\lambda_1^s-\lambda_2^c & -\lambda_3 \\ \lambda_3 & \omega_2\lambda_1^s+\lambda_2^c \end{bmatrix}\begin{Bmatrix}\beta_0\\ \bar{\beta}_0\end{Bmatrix} = 0, \tag{8.10}$$

where

$$\lambda_1^s = -kK_1'\,d^s+iK_1\,a^s,$$

$$\lambda_2^c = kK_1'\,e^c+iK_1\,b^c,$$

$$\lambda_3 = kK_1'(f^s+f^c)+iK_1(c^s+c^c).$$

In order to have a non-trivial solution for β_0 and $\bar{\beta}_0$, the correction to the frequency ω_2 must equal

$$\omega_2^2 = -\left[\left(\frac{\lambda_3}{\lambda_1^s}\right)^2 - \left(\frac{\lambda_2^c}{\lambda_1^s}\right)^2\right] = -[\gamma_3^2 - \gamma_2^2], \tag{8.11}$$

where, as we shall see, γ_3 and γ_2 are real.

Before presenting numerical results for γ_3 and γ_2, we note that although $\ln\epsilon$ has been treated as $0(1)$ in the analysis thus far, it is now of interest to show the explicit dependence on $\ln\epsilon$. It turns out that only λ_3 contains $\ln\epsilon$ from the effects of strain. We, therefore, write

$$\gamma_3 = \gamma_{31}^s \ln\frac{8}{\epsilon} + \gamma_{32}^c + \gamma_{32}^s \tag{8.12}$$

to show explicitly the effects of strain and curvature. Then, the unstable root ω_2 can be written

$$\omega_2 = -\mathrm{i}\left[\left(\gamma_{31}^s \ln\frac{8}{\epsilon} + \gamma_{32}^c + \gamma_{32}^s\right)^2 - (\gamma_2^c)^2\right]^{\frac{1}{2}}. \tag{8.13}$$

The flow will be unstable as long as ω_2 has a negative imaginary part. In the limit $\epsilon \to 0$, $\ln\epsilon$ will dominate and the flow will be unstable.

The numerical results for two specific cases will be presented in § 9 in comparison with experimental results.

9. Numerical results and discussion

The amplification rate for the instability of bending waves on a vortex ring is given by (8.13). This formula was derived for a core of constant vorticity without axial flow. For this case, the bending waves which have $\omega_0 = 0$ have special significance in the stability analysis and only this case has been investigated. The effects of strain and curvature have been separated by so grouping the non-homogeneous forcing terms in the governing equations (§8) and following them through the analysis.

Numerical results have been obtained for two cases: $k = 2.504$ and $k = 4.35$. These are crossing points of the dispersion relations at $\omega_0 = 0$ for the second and third radial mode (see figure 4).

The numerical results for the quantities appearing in (8.13) for these two cases are given in table 1. The results for these two wavenumbers are remarkably similar. The effects of curvature are much smaller than that of strain (γ_{32}^c is less than 15 % of γ_{32}^s and γ_2^c has negligible effect on ω_2).

Table 1. Numerical values for quantities appearing in (8.13)

	$k = 2.5$		$k = 4.35$	
γ_{31}^s	0.428		0.427	
γ_{32}^s ⎫ $\gamma_{32}^s + \gamma_{32}^c$	−0.534	−0.4549	−0.521	−0.48509
γ_{32}^c ⎭	0.0788		0.0357	
γ_2^c	0.3367		0.3409	

It is interesting to note that the $O(\ln\epsilon)$ amplification rate γ_{32}^s turns out to be three quarters that for a line vortex in a two-dimensional straining flow (Widnall & Tsai 1976). This is because the $O(\ln\epsilon)$ mean flow is a plane strain of magnitude $\frac{3}{4}$ and the $O(\ln\epsilon)$ terms do not couple with any of the terms arising from the scaled coordinates.

In dimensional form, the amplification rate of the instability is

$$\alpha = (\Gamma/2\pi R^2)\ (\omega_2). \tag{9.1}$$

It is convenient to use $\Gamma/4\pi R^2$ as a non-dimensional time for the vortex ring rather than $\Gamma/2\pi R^2$, which has been used up to now since it is convenient for the line filament; the amplification rate non-dimensionalized by $\Gamma/4\pi R^2$ is then

$$\overline{\alpha} = 2\omega_2.$$

This choice of non-dimensional amplification rate is consistent with the previous works of Widnall & Sullivan (1973) and Widnall *et al.* (1974).

For the second radial bending mode ($k = 2.5$) the non-dimensional amplification rate is then (from (8.13) with numerical values from table 1)

$$\overline{\alpha} = [(0.856 \ln (8/\epsilon) - 0.9102)^2 - 0.1138]^{\frac{1}{2}}. \tag{9.2}$$

For small ϵ, the last term in negligible and

$$\overline{\alpha} \simeq 0.856 \ln (8/\epsilon) - 0.9102. \tag{9.3}$$

Both (9.2) and (9.3) are positive for $\epsilon < 1$.

The work of Widnall & Sullivan (1973) introduced the spatial amplification factor

$$\alpha_x = \overline{\alpha}/\tilde{V},$$

where $\tilde{V}$ is the propagation velocity of the vortex ring nondimensionalized by $\Gamma/4\pi R$. For a thin ring, $\tilde{V}$ depends only upon the distribution of vorticity within the core. For constant vorticity

$$\tilde{V} = \ln (8/\epsilon) - \tfrac{1}{4}$$

so that the theoretical result for spatial amplification of the waves is

$$\alpha_x = 0.856 - 0.694/\tilde{V}. \tag{9.4}$$

It is most likely coincidental that this result turns out to be numerically close to the original results presented by Widnall & Sullivan (1973) for the spatial amplification rate of the instability of the vortex ring using the Biot-Savart law and the cut-off method.

The predictions of this analysis for the wavelength of the vortex ring instability are somewhat indirect and may be stated as follows: The vortex ring is unstable to specific wavenumbers for which waves on a line filament with the same vorticity distribution would be stationary ($\omega_0 = 0$). These wavelengths are proportional to core size so that ka is a constant κ for these waves. ($\kappa = 2.5$, 4.35, etc., for constant vorticity; the first value is generally observed experimentally.) For a given core size, a the wavelength λ is then determined. If an integer number of these waves will fit on the ring, the ring will be unstable to that number of waves. For a given ϵ, $n = \kappa/\epsilon$.

If the observed value n is used to diagnose the properties of the ring as in the experiments of Maxworthy (1976), then $\tilde{V} = \ln (8n/\kappa) - \frac{1}{4}$ and

$$\alpha_x = 0.856 - \frac{0.694}{\ln (8n/\kappa) - \frac{1}{4}}. \tag{9.5}$$

For the line filament in strain, and likely also for the ring, there is actually a small band of wavenumbers centred on the critical wavenumber which will be amplified (Tsai & Widnall 1976). If these bands overlap slightly, the ring will always be unstable. The stability criterion $\omega_0 = 0$ was used previously in Widnall *et al.* (1974) who obtained good agreement between the measured and predicted number of waves in the instability of a given ring; figure 5 is reproduced from their paper.

The only measurements of an amplification rate are those of Sullvian (Widnall & Sullivan 1973). The difficulty of comparing the predictions of the theory with experimental results is considerable. The only case for which the vorticity distribution, the circulation, the induced velocity, the amplification rate and the number of waves in the unstable mode were measured is that presented by Widnall & Sullivan (1973) (see also Sullivan, Widnall & Ezekiel (1973) for the ring of figure 1 which displayed an instability at $n = 7$). The vorticity in the core was not constant. This ring had a measured induced velocity of $\tilde{V} = 2.46$ but had a $\tilde{V} = 3.01$ as calculated from direct measurement of the vorticity distribution. This ring was quite fat, corresponding roughly to $\epsilon = 0.3$. The measured amplification rate for this ring was $\alpha_x = 0.69$. If $\tilde{V}$ from the measured vorticity distribution is used in (9.4), then $\alpha_x = 0.625$; if $\tilde{V}$ from the observed n is used, then $\alpha_x = 0.612$.

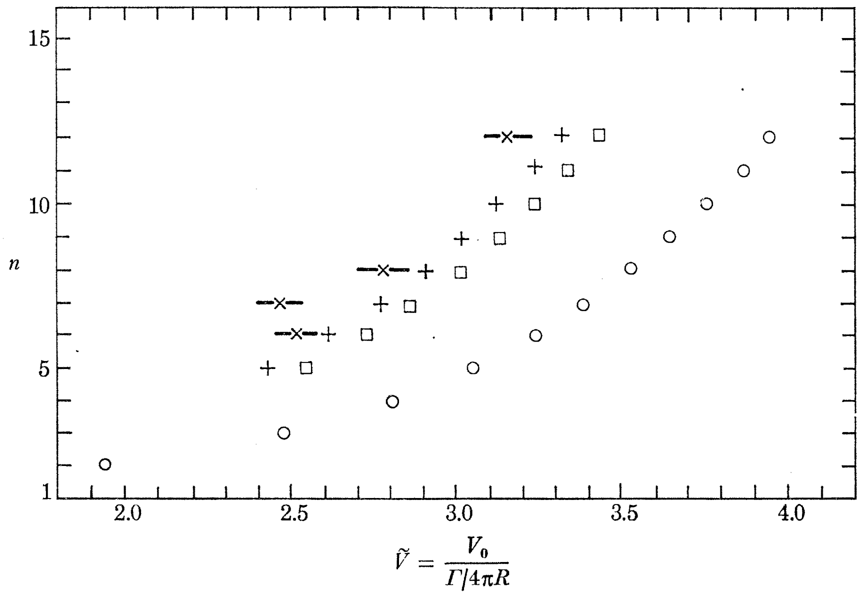

FIGURE 5. Theoretical and experimental results for the value of $\tilde{V}$ for which a given mode n is unstable. □, Constant vorticity $\kappa = 2.5$; +, continuous vorticity $\kappa = 2.7$; ○, asymptotic long-wave result $\kappa = 1.44$; —×—, experiment (Widnall & Sullivan 1973).

Thus, the theory is in reasonable agreement with experiment for both the number of waves (Widnall *et al.* 1974) and the amplification rate for the observed instability of the vortex ring. There have been many other observations of this instability (see Maxworthy 1976), but no additional complete measurements of the instability parameters have been reported.

The theory presented here is valid only for a thin vortex ring, of constant vorticity in an inviscid flow. It shows that these vortex rings are unstable to a mode consisting of an integer number of bending waves of certain critical wavelengths proportional to core radius. Vortex rings are believed to persist in two states that are not covered by this theory: at low Reynolds number and with a turbulent core. At low Reynolds numbers, growth of the core with time could continually keep the ring from remaining unstable to a given mode long enough to be noticeably amplified. The self-preservation of the ring with a turbulent core remains a mystery. Maxworthy (1976) has suggested that axial flows are created within the core and stabilize the ring, but since the theory developed to date does not include the effects of axial velocity, no definitive statement can be made about this possibility.

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FIGURE 1. Flow visualization of the vortex ring instability; $n = 7$. Taken from Widnall & Sullivan (1973).

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